Mathematics Specialised

Course Code: MTS415114

Students tended to find this exam more difficult than the exam last year. However the length of the exam was appropriate for most students.

Students are reminded to complete each section in a separate booklet. While this was less of an issue than last year, it still occurred in some cases.

Generally students performed well on the induction question and question 6 in this section. Weaker students tended to be more successful in these two questions than the first one. Overall most students knew the process required for each question well.

Question 1

Many students did not recognise this question as a Geometric Series and thus gained no marks for this question. For those that did, most gave the correct answer although some students gave the incorrect values for a and r. It is definitely helpful to write out the first few terms of the series to overcome this error.

Question 2

Students were much more successful with this question than Q1. Some students unsuccessfully tried to use the formula $U_n = S_n - S_{n-1}$ here. Most students recognised that the series summation formulae could be used. Many students left obvious cancellation or unfactorised terms in their answer and were penalised for this.

Question 3

Almost all students were comfortable with the process required for this question but made errors along the way. It is important for students to remember that $|a_n - L|$ must be put over a common denominator first, then work to remove the absolute value signs can be done. A subtle error here was saying that $|n - n^2| < \varepsilon \Leftarrow |n| < \varepsilon$ by "rigging" when in fact this was making the magnitude of the LHS smaller, as for example if $n = 10$ this is reducing the magnitude of the numerator from 90 to 10.

In removing the absolute value signs a number of students simplified $\left|\frac{-n^2+n}{n^3-1}\right|$ to be $\frac{n^2+n}{n^3-1}$, whereas a common factor of -1 needed to be taken out of the numerator resulting in $\frac{n^2-n}{n^3-1}$. Once $\frac{n^2-n}{n^3-1}$ was arrived at, many more students tried to "rig" the question by simplifying this to $\frac{n^2}{n^3}$. Whilst this was correct for the numerator as the LHS was being made larger, the denominator was also being made larger which was incorrect.

A pleasing number of students knew the difference of cubes formula. Generally the proof was finished off well.

Question 4

This question was quite well done overall with most students handling the algebra without difficulty. With the conclusion of the proof it is important to state that because both P_1 is true and that P_k is true implies P_{k+1} is true the proof is true by Mathematical Induction. Many students took too long in writing a conclusion that was more verbose than necessary. It is worthwhile students practising writing the minimum that needs to be stated in exams.

Question 5

(a) By far the easiest way to do this question was to start with the RHS and find a common denominator. A large number of students used partial fractions on the LHS instead, which while being successful in most cases, took much longer to arrive at an answer. Some students used the $\frac{last-first}{difference}$ approach without emphasising that they were multiplying by $\frac{2}{2}$, this was not sufficient for a proof.

(b)(i): Many students answered this question successfully. Common problems were not noticing that the summation started from $r = 2$ rather than $r = 1$ and often generating a term of $\frac{1}{0}$ which was interpreted in a variety of different ways! A number of students left out this first term and so arrived at $\frac{1}{2}$ instead of $\frac{3}{4}$ as the limit. When students are using the "V_r" notation they should be careful to spell out the definition of V_r to avoid confusion.

(b)(ii): Students answered this question very well on the whole, this question was given full marks depending on the answer they gave in part (i).

Question 6

(a) A number of students did not know to take $\frac{n}{2}$ terms for each series, despite it being stated in bold that n was even. Almost all students were able to correctly identify the two summations and use their formulae correctly. Some marks were lost in many cases for not providing a fully factorised answer which combined the two summations. On the whole this question was well done.

(b) Most students simply substituted $n = 97$ into their formula and so received no marks for this part. Recognising that the formula could only be used for even numbers received part marks, and achieving a correct answer based on their answer for part (a) was given full marks.

Section B: Matrices and Linear Transformations

Question 7

Most candidates knew that it was necessary to compute AB and BA, and show that they were not equal for the matrix multiplication not to commute.

Question 8

In general this question was answered well. However, there was a proportion of candidates who made simple algebraic errors when determining $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} - k\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, while there were a number of candidates who assumed that k was a matrix.

When unpacking a question, it is important to look at any literals in the context of the content that is being examined in the section to be answered: the nomenclature for matrices is upper case letters of the alphabet, while scalars use lower case.

Question 9

(a) Most candidates were able to interpret the given definition of idempotent matrices, and, as a consequence, answered this part successfully.

(b) Although singular was highlighted in the wording of the question in bold, many candidates did not use this fact, and hence were unable to successfully complete the question.

Question 10

(a) A linear function which passes through the origin may be expressed in the form $y = (\tan\theta)x$, where the line makes an angle of θ with the positive direction of the x-axis. Those candidates who were unfamiliar with this fact could see no point in recalling $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ and resorted to half-angle formulae, or using vertical rise over horizontal run (to find $\sin\theta$ and $\cos\theta$, and thence $\cos 2\theta$ and $\sin 2\theta$), or nothing at all.

(b) The majority of candidates were perplexed by the term midpoint and as a consequence were unable to develop the final conclusion. However, most candidates were able to determine the images of the extremities of the given interval.

Question 11

(a) Some students provided succinct and clear solutions in obtaining the required result. However, there were many candidates who fudged steps along the way to obtain the required result: the combination of the process of Gauss-Jordan elimination and algebraic fractions proved to be their nemesis.

Furthermore, it is important that the marker is able to follow each candidate's process, and the sub-steps along the way. It is also important for candidates to be consistent when annotating operational steps for, in some cases, it was extremely difficult to decipher what a candidate had calculated.

(b) Many candidates missed the meaning of "least positive integral value of k" when unpacking the question, and, as a consequence, found the value of k which yielded the least positive coordinates of the point of intersection of the three planes.

Question 12

(a) Most candidates successfully found the reflection and rotation matrices, and hence the composition matrix although there were the few who incorrectly found the clockwise rotation matrix, and/or the composition matrix by using the incorrect order of operations.

(b) The majority of candidates answered this well (with the results obtained in part a).

(c) Few candidates were able to determine the value of φ, and it seemed to be the form of the linear equation that caused confusion in the first instance.

Section C Differential Calculus, Areas and Volumes

Overall this section was done well by most students. There were plenty of 'C' and 'B' standard questions. Q17 was a very hard question and only very able students made much progress with it.

Question 13

A straight forward Maclaurin Series question with many students getting full marks. For those making errors the common mistake was missing out the negative sign when finding the derivative of 1-x .

Question 14

A straight forward differentiation question with many students getting full marks. Errors were mostly made when solving for a rather than in the differentiating.

Question 15

An implicit differentiation question that most, but not all, did well. A common error was not to find the derivative of 1 to be zero, which made the rest of the algebra more difficult than it should have been. Also some students found it difficult to find the value of y when $= \frac{1}{4}$.

Question 16

This question was answered using a couple of different substitutions, though most used $u = 2 - x$ successfully.A calculator only solution without using a substitution as directed got only $1\frac{1}{2}$ out of 5 marks.

Some students did not use a substitution method as the question directed but instead used Integration by Parts. Because this made the question harder full marks were given for this approach, though it is recommended that students follow directions.

Question 17

Only 9 students found the correct solution to this question.

It was too hard for most students to visualise which combination of volumes should be used to find the required volume. Also, the idea of separating the parabola into two parts $x = 1 + \sqrt{y}$ and $x = 1 - \sqrt{y}$ was beyond almost all students.

Some students found the volume around the x axis, not the y axis as the question asked. Because this made the question much easier and eliminated all the difficult visualising of volumes required, this approach received very few marks.

Question 18

This question was done very well. Most students explained their working and justified their choice of the nature of all required points.

Unfortunately some did not get this far, perhaps due to lack of time, and so missed a very comfortable 7 marks.

Section D - Integral Calculus

Question 19

This question was reasonably well done. Most students realised that $tanx = \frac{sinx}{cosx}$ and let $u = cosx$.

Question 20

The majority of students were successful with this question. However, a surprising number of students were unable to complete the square.

Question 21

Marks were generously awarded. Many students were unable to make any real progress with this question. Those that attempted to substitute for $\sqrt{x}$ or for $1 + \sqrt{x}$ were rewarded. Some candidates did use the graphics calculator to find the integral. Those that did this and then found the value by substituting values did get some credit.

Question 22

The majority of students started this question well by applying integration by parts correctly. However, a significant number made no progress past this first stage. Those that separated the fractions formed in the second stage were very successful (for example, $\frac{\frac{x^2}{2}+x}{1+x^2} = \frac{x^2+2x}{2(x^2+1)} = \frac{1}{2} + \frac{2x}{2(x^2+1)} - \frac{1}{2(1+x^2)}$).

Question 23

Very pleasingly, this question was very well done by the majority of students. Homogeneous equations were generally well understood and completed fully by many students.

Question 24

This question was poorly done.

Many students did not realise that this was a separable D.E. The use of g and c as constants caused great confusion leading to difficulty rearranging the equation appropriately.

Those students that managed to separate the equation successfully made reasonable progress. Many forgot to add the constant of integration which made it impossible to reach the correct answer. A number had put +c after integrating and then rejected it since c was already there!

Section E Complex Numbers

Questions 25 – 27

It was impossible to judge whether students were fatigued or discouraged for the last section but these 3 mark questions did not provide the opportunity to gain 9 marks and set themselves up for a successful section. On the contrary many students accumulated few if any marks and did not make any coherent headway with any of these questions.

Question 25

Many students found this question extremely confusing with too many adjustments to consider. Few students managed both parts successfully.

Arg $4z^3$ = Arg z^3 = 3Arg z = 3θ ;
$|4z^3|$ = $4|z^3|$ = $4r^3$

Question 26

Many students became lost in the sequential logic and made many simple numerical errors
If z = a + bi then Im(z) = I
Now z = a + i and z − 2i = a − i .
Given that $|z - 2i|$= 3
Now $|a - i|$= 3 $\Rightarrow \sqrt{a^2+1}$= 3 .
Squaring both sides gives
$a^2 + 1$= 9,
$\Rightarrow a^2$= 8
Thus a = $\pm\sqrt{8}$

Question 27

Students stumbled onto the correct solutions although their recording sequence demonstrated that they found the question to be a little ambiguous, expecting complex and linear factors. The end result was the same.

Question 28

(a) Students were mostly able to correctly deduce the correct base angle $\frac{\pi}{4}$; and find the 1st solution $\frac{\pi}{16}$ despite some strange applications of De Moivre's Theorem. Diagrams were not neat but mostly demonstrated an understanding of equally spaced solutions.
(b) Some students expected part (b) to be related to part (a) – this was not the case. Some students did not even 'see' part (b). Of the rest many students were able to prove the expression.

Question 29

This question was generally well done. A variety of techniques of solution were used, some being more efficient than others. See the solutions for the recommended method. Care is needed to not make errors in the early stages to avoid making life harder later on. Some students did not seem to know the difference between a factor and a solution. Students are asked to “show the method used”. Thus some comment is needed in obtaining the $z^2 - 2z + 26$ factor and some working is needed to factorise this expression.

Question 30

Parts (a) and (b) (i) were generally well done. There was some confusion as to what was required in part (a). Two separate regions were expected to be illustrated on the one Argand diagram. Students who interpreted the question differently were usually able to show sufficient evidence to gain full marks for that part. Very few students made any progress on part (b) (ii).

Section A

Q1: $\sum_{r=1}^{n} 2^{-r}$ is a GP with $a = 2^{-1} = \frac{1}{2}$ and $r = \frac{2^{-2}}{2^{-1}} = 2^{-1} = \frac{1}{2}$.

Q2: $\sum_{r=1}^{n} r(3r-1) = \sum_{r=1}^{n} \left(3r^2 - r\right) = 3\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r =$

$\frac{3n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n+1)[2n+1-1]}{2} = n^2(n+1).$

Q3: $\left\{\frac{n^3-n^2+n-1}{n^3-1}\right\}$ converges to 1 if given any $\varepsilon > 0 \; \exists \; N(\varepsilon)$ s.t. $\left|\frac{n^3-n^2+n-1}{n^3-1} - 1\right| < \varepsilon$

$\Leftarrow \left|\frac{n^3-n^2+n-1}{n^3-1} - 1\right| < \varepsilon$

$\Leftarrow \left|\frac{n^3-n^2+n-1-(n^3-1)}{n^3-1}\right| < \varepsilon$

$\Leftarrow \left|\frac{-n^2+n}{n^3-1}\right| < \varepsilon$

$\Leftarrow \frac{n^2-n}{n^3-1} < \varepsilon$, since $n - n^2 < 0$ and $n^3 - 1 > 0$ for n large enough.

$\Leftarrow \frac{n^2-n}{n^3-n^2} < \varepsilon$

since $n^3 - n^2 < n^2 - 1$

for n large enough and $n^3 - n^2 > 0$.

$\Leftarrow \frac{n^2-n}{n(n^2-n)} < \varepsilon$

$\Leftarrow \frac{1}{n} < \varepsilon$

$\Leftarrow n > \frac{1}{\varepsilon}$

$\therefore$ choose $N = \frac{1}{\varepsilon}$

$\therefore \left\{\frac{n^3-n^2+n-1}{n^3-1}\right\}$ converges to 1.

OR

$\Leftarrow \frac{n(n-1)}{(n-1)(n^2+n+1)} < \varepsilon$

$\Leftarrow \frac{n}{n^2+n+1} < \varepsilon$

$\Leftarrow \frac{n}{n^2} < \varepsilon$, since $\frac{n}{n^2+n+1} < \frac{n}{n^2}$

$\Leftarrow \frac{1}{n} < \varepsilon$

Q4: Prove that $2 + 3.2 + 4.2^2 + 5.2^3 + \cdots + (n+1)2^{n-1} = n.2^n$ for all $n \in N$.

Let P_n be $2 + 3.2 + 4.2^2 + 5.2^3 + \cdots + (n+1)2^{n-1} = n.2^n$ for all $n \in N$.

Prove P_1 is true:

LHS = 2 RHS = 1.2^1

$= 2 =$ LHS

$\therefore P_1$ is true.

Assume P_k is true.

ie. $2 + 3.2 + 4.2^2 + 5.2^3 + \cdots + (k+1)2^{k-1} = k.2^k$ Equation (*)

RTP P_{k+1} is true.

ie. $2 + 3.2 + 4.2^2 + 5.2^3 + \cdots + (k+1)2^{k-1} + (k+2)2^k = (k+1).2^{k+1}$

LHS =

$= 2 + 3.2 + 4.2^2 + 5.2^3 + \cdots + (k+1)2^{k-1} + (k+2)2^k$

$= k.2^k + (k+2)2^k$ Using (*)

$= k.2^k + k.2^k + 2.2^k$

$= 2k.2^k + 2^{k+1}$

$= k.2^{k+1} + 2^{k+1}$

$= (k+1)2^{k+1}$

= RHS

$\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true

$\therefore$ by the principle of mathematical induction, P_n is true for all $n \in N$.

$\therefore$ $2 + 3.2 + 4.2^2 + 5.2^3 + \cdots + (k+1)2^{k-1} = k.2^k$ for all $n \in N$.

Q5: (a) RHS = $\frac{1}{2}\left(\frac{1}{r-1} - \frac{1}{r+1}\right) = \frac{1}{2}\left(\frac{(r+1)-(r-1)}{(r-1)(r+1)}\right) = \frac{1}{2} \times \frac{2}{(r-1)(r+1)} = \frac{1}{r^2-1}$ = LHS

(b) (i) Let $U_r = \frac{1}{r^2-1} = \frac{1}{2}\left(\frac{1}{r-1} - \frac{1}{r+1}\right) = \frac{1}{2}(V_r - V_{r+2})$ where $V_r = \frac{1}{r-1}$.

$\therefore \sum_{r=2}^{n} \frac{1}{r^2-1} = \sum_{r=2}^{n} U_r = \frac{1}{2}\sum_{r=2}^{n}(V_r - V_{r+2})$

$= \frac{1}{2}\big((V_2 - V_4) + (V_3 - V_5) + (V_4 - V_6) + \cdots + (V_{n-2} - V_n) + (V_{n-1} - V_{n+1}) + (V_n - V_{n+2})\big)$

$= \frac{1}{2}(V_2 + V_3 + \cdots - V_{n+1} - V_{n+2})$

$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2} \times \frac{\frac{3}{2}n(n+1)-(n+1)-n}{n(n+1)} = \frac{3n(n+1)-2(n+1)-2n}{4n(n+1)} = \frac{3n^2-n-2}{4n(n+1)}$.

(ii) $\frac{3n^2-n-2}{4n(n+1)} = \frac{3n^2-n-2}{4n^2+4n} \div \frac{n^2}{n^2} = \frac{3-\frac{1}{n}-\frac{2}{n^2}}{4+\frac{4}{n}}$ and so $\sum_{r=2}^{\infty} \frac{1}{r^2-1} = \frac{3}{4}$ as $\frac{3-\frac{1}{n}-\frac{2}{n^2}}{4+\frac{4}{n}} \to \frac{3}{4}$ as $n \to \infty$.

Q6: (a) $1 + 2^2 + 3 + 4^2 + 5 + 6^2 + \cdots$

$= 1 + 3 + 5 + \cdots \left(to\ \frac{n}{2}\ terms\right) + 2^2 + 4^2 + 6^2 + \cdots \left(to\ \frac{n}{2}\ terms\right)$ since n is even.

$= \sum_{r=1}^{n/2}(2r-1) + \sum_{r}^{n/2}(2r)^2 = 2\sum_{r=1}^{n/2}(r) - \sum_{r=1}^{n/2}(1) + 4\sum_{r=1}^{n/2} r^2$

$= 2\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2} - \frac{n}{2} + 4\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)\left(2\frac{n}{2}+1\right)}{6} = \frac{n}{2}\left(\frac{n}{2}+1\right) - \frac{n}{2} + \frac{\frac{n}{2}(n+2)(n+1)}{3}$

$= \frac{n^2}{4} + \frac{n}{2} - \frac{n}{2} + \frac{n(n+1)(n+2)}{6} = \frac{3n^2+2n(n+1)(n+2)}{12} = \frac{n(n+4)(2n+1)}{12}$.

(b) Sum of 96 terms: sum = $\frac{96(96+4)(2\times96+1)}{12}$ = 154400.

Term 97 is the 49th term in the second sequence which is $2\times49-1=97$.

$\therefore$ the sum to 97 terms is 154497.

Section B

Question 7 **(3 marks)**

Show that for matrices $A=\begin{pmatrix}2&3\\1&4\end{pmatrix}$ and $B=\begin{pmatrix}-1&2\\-3&3\end{pmatrix}$, matrix multiplication is not commutative.

We have $A\times B=\begin{pmatrix}2&3\\1&4\end{pmatrix}\times\begin{pmatrix}-1&2\\-3&3\end{pmatrix}=\begin{pmatrix}-13&13\\-13&14\end{pmatrix}$;

but also $B\times A=\begin{pmatrix}-1&2\\-3&3\end{pmatrix}\times\begin{pmatrix}2&3\\1&4\end{pmatrix}=\begin{pmatrix}0&5\\-3&3\end{pmatrix}$.

Since $A\times B\neq B\times A$ we see that matrix multiplication is not generally commutative.

Note: a calculator can be used here, and it suffices to calculate just one comparable element in each matrix AB and BA.

Question 8 **(3 marks)**

Find all values of *k* for which the matrix $\begin{pmatrix}1&2\\0&1\end{pmatrix}-k\begin{pmatrix}1&0\\2&1\end{pmatrix}$ has an inverse.

We have $\begin{pmatrix}1&2\\0&1\end{pmatrix}-k\begin{pmatrix}1&0\\2&1\end{pmatrix}=\begin{pmatrix}1&2\\0&1\end{pmatrix}-\begin{pmatrix}k&0\\2k&k\end{pmatrix}=\begin{pmatrix}1-k&2\\-2k&1-k\end{pmatrix}$, and so we require that the determinant is non-zero. That is, that $(1-k)^2+4k\neq0$.

That is,

$$\begin{aligned}1-2k+k^2+4k&\neq0\\k^2+2k+1&\neq0\\(k+1)^2&\neq0\\k&\neq-1\end{aligned}$$

So provided $k\neq-1$ the inverse will exist.

Question 9 **(5 marks)**

A 2×2 matrix P is called *idempotent* if $P^2 = P$.

(a) Show that the matrix $\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$ is idempotent.

$$\begin{aligned}\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}^2 &= \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \times \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 9-6 & 18-12 \\ -3+2 & -6+4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\end{aligned}$$

(b) Show that all **singular** 2×2 matrices of the form $\begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix}$ are idempotent.

We calculate

$$\begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix}^2 = \begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+bc & b \\ c & bc+\frac{1}{4} \end{pmatrix}$$

Since the matrix is singular, we have $\det\begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix} = 0$, which is to say that $bc = \frac{1}{4}$.

We therefore have

$$\begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix}^2 = \begin{pmatrix} \frac{1}{4}+\frac{1}{4} & b \\ c & \frac{1}{4}+\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix}$$

Thus we see that all singular matrices of the form $\begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix}$ are idempotent.

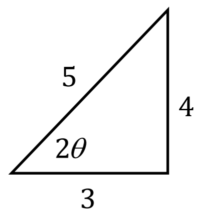
Question 10 **(6 marks)**

(a) Recalling $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, show that the matrix of a reflection in the line $y = \dfrac{1}{2}x$ is $\dfrac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

The reflection matrix is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, where we are given $\tan\theta = \dfrac{1}{2}$.

We employ $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ to obtain $\tan 2\theta = \dfrac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} = \dfrac{4}{3}$.

Pythagoras' Theorem then lets us find $\cos 2\theta$ and $\sin 2\theta$:



$$\sin 2\theta = \frac{4}{5} \text{ and } \cos 2\theta = \frac{3}{5}.$$

The reflection matrix, therefore, is

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

(b) Show that under reflection in the line $y = \dfrac{1}{2}x$, the midpoint of the line segment joining $(10, 10)$ to $(20, 0)$ is mapped to the midpoint of the images of $(10, 10)$ and $(20, 0)$.

The midpoint of the line segment joining $(10, 10)$ to $(20, 0)$ is $(15, 5)$.

The images of $(10, 10)$, $(20, 0)$ and $(15, 5)$ are found:

$$\frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 10 & 20 & 15 \\ 10 & 0 & 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 70 & 60 & 65 \\ 10 & 80 & 45 \end{pmatrix} = \begin{pmatrix} 14 & 12 & 13 \\ 2 & 16 & 9 \end{pmatrix}.$$

That is, $(14, 2)$ and $(12, 16)$, with a midpoint image of $(13, 9)$

The points $(14, 2)$ and $(12, 16)$ have a midpoint given by $\left(\dfrac{14+12}{2}, \dfrac{2+16}{2}\right) = (13, 9)$, so we see that the image of the midpoint is indeed the midpoint of the images.

Question 11 **(6 marks)**

The 3×3 system of equations

$$\begin{aligned} 6x+2y-10z&=0 \\ 4y-8z&=12 \\ 14x+7y-12z&=k \end{aligned}$$

where $k \in Z^+$, describes three planes that intersect in a single point.

(a) Use Gauss-Jordan elimination to show that the system reduces to the augmented matrix with form

$$\left(\begin{array}{ccc:c} 1 & 0 & 0 & \dfrac{17+k}{8} \\ 0 & 1 & 0 & \dfrac{k-23}{16} \\ 0 & 0 & 1 & \dfrac{k-7}{16} \end{array}\right)$$

(b) Determine the least positive integral value of *k* such that the three coordinates of the intersection point are integers. State the point of intersection.

The system is

$$\begin{aligned} 6x+2y-10z&=0 \\ 4y-8z&=12 \\ 14x+7y-12z&=k \end{aligned}$$

This gives the augmented matrix which may be developed as follows:

$$\left(\begin{array}{ccc|c} 6 & 2 & -10 & 0 \\ 0 & 4 & -8 & 12 \\ 14 & 7 & -12 & k \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & 1 & -5 & 0 \\ 0 & 1 & -2 & 3 \\ 14 & 7 & -12 & k \end{array}\right) \begin{array}{l} R_1/2 \\ R_2/4 \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 3 & 1 & -5 & 0 \\ -3 & 0 & 3 & 3 \\ -7 & 0 & 23 & k \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 7R_1 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 3 & 1 & -5 & 0 \\ 1 & 0 & -1 & -1 \\ -7 & 0 & 23 & k \end{array}\right) \begin{array}{l} \\ R_2/(-3) \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 0 & 1 & -2 & 3 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 16 & k-7 \end{array}\right) \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 + 7R_2 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 0 & 1 & -2 & 3 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & \frac{k-7}{16} \end{array}\right) \begin{array}{l} \\ \\ R_3/16 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 0 & 1 & 0 & \frac{17+k}{8} \\ 1 & 0 & 0 & \frac{k-23}{16} \\ 0 & 0 & 1 & \frac{k-7}{16} \end{array}\right) \begin{array}{l} R_1 + 2R_3 \\ R_2 + R_3 \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{k-23}{16} \\ 0 & 1 & 0 & \frac{17+k}{8} \\ 0 & 0 & 1 & \frac{k-7}{16} \end{array}\right)$$

For Part (b) we require the least positive *k* such that the three numbers, $\frac{17+k}{8}$, $\frac{k-23}{16}$ and $\frac{k-7}{16}$ are integral. Setting $k=7$ does the job, giving the point of intersection

$$\begin{pmatrix} \frac{17+7}{8} \\ \frac{7-23}{16} \\ \frac{7-7}{16} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}.$$

Question 12 **(7 marks)**

An ellipse is subjected to a reflection in the line $y = x$, followed by an anticlockwise rotation through $\frac{\pi}{4}$ radians.

(a) Determine the matrix that represents the composition of these two transformations.

The reflection matrix (noting that $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$) is $\begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The rotation matrix is $\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

The composition is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.

(b) The image of the source ellipse under the composition of these two transformations is itself an ellipse with equation $\frac{x^2}{9} + y^2 = 1$.

Determine the equation of the source ellipse in a form with integer coefficients.

The equation of the image ellipse is $\frac{(x')^2}{9} + \frac{(y')^2}{1} = 1$.

Under the transformation, $x' = -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$ and $y' = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$.

The source ellipse, therefore, has equation

$$\frac{\left(-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{9} + \frac{\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{1} = 1.$$

This reduces as follows:

$$\frac{\frac{x^2}{2} - xy + \frac{y^2}{2}}{9} + \frac{\frac{x^2}{2} + xy + \frac{y^2}{2}}{1} = 1$$

$$\frac{x^2}{2} - xy + \frac{y^2}{2} + 9\left(\frac{x^2}{2} + xy + \frac{y^2}{2}\right) = 9$$

$$5x^2 + 8xy + 5y^2 - 9 = 0$$

(c) This composition is equivalent to a reflection in the line $y = x\tan(\varphi)$. Determine a value of φ.

The reflection matrix is $\begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. This means $\cos 2\varphi = -\frac{1}{\sqrt{2}}$ and $\sin 2\varphi = \frac{1}{\sqrt{2}}$, and so $2\varphi = \frac{3\pi}{4}$ [we are in Quadrant 2], and thus $\varphi = \frac{3\pi}{8}$ is one possible value.

Section C

Question 13 **(3 marks)**

Write the Maclaurin series expansion for $\frac{1}{1-x}$ as far as the term in x^3.

$\frac{1}{1-x} = 1 + x + x^2 + x^3$, to the term in x^3.

Q14: If $y = \arctan x$, find the value of *a* given that $\frac{d^2y}{dx^2} = ax\left(\frac{dy}{dx}\right)^2$, for all *x*.

$\frac{dy}{dx} = \frac{1}{1+x^2}$. $\frac{d^2y}{dx^2} = -1 \times (1+x^2)^{-2} \times 2x = -2x\left(\frac{1}{1+x^2}\right)^2 = -2x\left(\frac{dy}{dx}\right)^2$.

So $a = -2$.

Q15: A curve is defined implicitly by $\sqrt{x} + \sqrt{y} = 1$.

Find the equation of the tangent to the curve at the point with $x = \frac{1}{4}$.

$\sqrt{x} + \sqrt{y} = 1$

$\therefore \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \times \frac{dy}{dx} = 0$

Substitute $x = \frac{1}{4}$ and $y = \frac{1}{4}$

$\frac{1}{2} \times \frac{1}{\frac{1}{2}} + \frac{1}{2} \times \frac{1}{\frac{1}{2}} \times \frac{dy}{dx} = 0$

$\therefore 1 + \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -1$

Equation of tangent: $y - \frac{1}{4} = -1\left(x - \frac{1}{4}\right)$

$\therefore y = -x + \frac{1}{2}$.

When $x = \frac{1}{4}$

$\sqrt{\frac{1}{4}} + \sqrt{y} = 1$

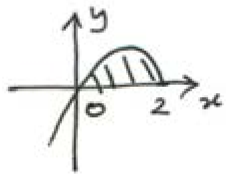
$\therefore \sqrt{y} = \frac{1}{2}$

$\therefore y = \frac{1}{4}$.

Question 16 **(5 marks)**

A curve is given by the equation $y = x\sqrt{2-x}$ for $x \le 2$.

Using a suitable substitution, evaluate an integral and hence find the exact area of the region enclosed by the curve and the *x*-axis between its zeros.



$y = x\sqrt{2-x}$

Area = $\int_0^2 x\sqrt{2-x}\,dx$

$= \int_0^2 (2-u)\sqrt{u} \times -du$ Let $u = x - 2$

$= \int_0^2 2u^{\frac{1}{2}} - u^{\frac{3}{2}} \times du$ $\therefore du = -dx$ and $x = 2 - u$

$= \left[2\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^2$

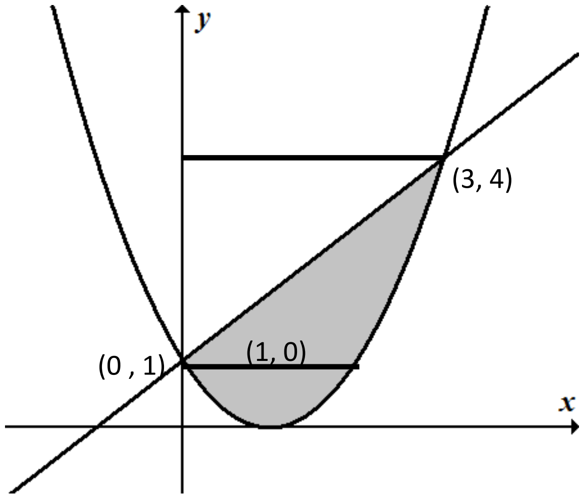
$= \frac{4}{3}2^{\frac{3}{2}} - \frac{2}{5}2^{\frac{5}{2}}$

$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$ units^2.

Q17:

$y = x + 1$ so $x = y - 1$.

$y = (x-1)^2$ so $x = \pm\sqrt{y} + 1$



$$\therefore Volume = \pi\left(\int_0^1 \left(\sqrt{y}+1\right)^2 dy - \int_0^1 \left(-\sqrt{y}+1\right)^2 dy + \int_1^4 \left(\sqrt{y}+1\right)^2 dy - \int_1^4 (y-1)^2\,dy\right)$$

$$= \pi\left(\int_0^4 \left(\sqrt{y}+1\right)^2 dy - \int_0^1 \left(-\sqrt{y}+1\right)^2 dy - \int_1^4 (y-1)^2\,dy\right) = \left(\frac{68}{3} - \frac{1}{6} - 9\right)\pi = \frac{27}{2}\pi\ units^3.$$

Q18: $f(x) = e^x - e^{1-x} - (e+1)x$. $\therefore f'(x) = e^x + e^{1-x} - (e+1)$

$\therefore f''(x) = e^x - e^{1-x}$

(a) Stationary Points occur when $f'(x) = 0$. $\therefore e^x + e^{1-x} = e + 1$. $\therefore x = 1$ *or* $x = 0$.

$f(0) = 1 - e$ and $f''(0) = 1 - e < 0$. So $(0, 1 - e)$ is a maximum.

$f(1) = -2$ and $f''(1) = e - 1 > 0$. So $(1, -2)$ is a minimum.

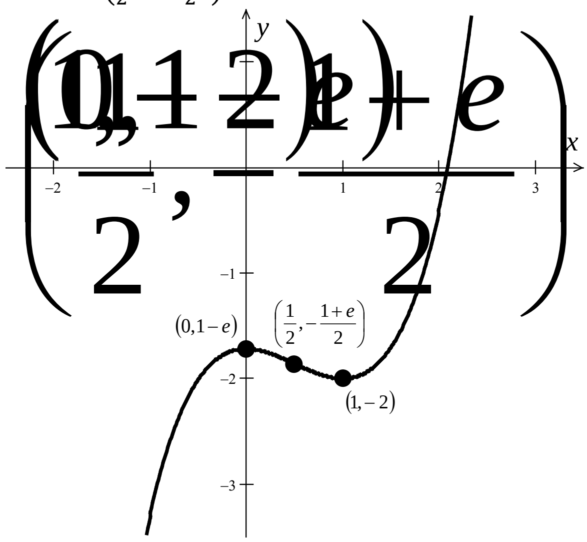
Points of inflection occur when $f''(x) = 0$ and $f'(x) \neq 0$.

$f''(x) = 0$ when $e^x - e^{1-x} = 0 \;\therefore\; e^x = e^{1-x} \;\therefore\; x = 1 - x \;\therefore\; x = \frac{1}{2}$. $f'\left(\frac{1}{2}\right) \neq 0$.

$f\left(\frac{1}{2}\right) = -\frac{1+e}{2}$ and $f''(x)$ changes in sign through $\frac{1}{2}$ since $f''(0) < 0$ and $f''(1) > 0$.

A point of inflection occurs at $\left(\frac{1}{2}, -\frac{1+e}{2}\right)$. Since $f'\left(\frac{1}{2}\right) < 0$ it is on a falling curve.

(b)



Section D

Q19: Consider $\int tanx dx = \int \frac{sinx}{cosx} dx$. Let $u = cosx$

$= -\int \frac{du}{u} = -ln|u| + C = -ln|cosx| + C$

$\therefore \int_0^{\frac{\pi}{3}} \tan x dx = -[ln|cosx|]_0^{\frac{\pi}{3}} = -ln\left|cos\frac{\pi}{3}\right| + ln|cos0| = -ln\frac{1}{2} + 0 = ln2.$

Q20: $\int \frac{1}{x^2+4x+5} dx = \int \frac{1}{x^2+4x+4+1} dx = \int \frac{1}{(x+2)^2+1} dx = arctan(x + 2) + C.$

Q21: Consider $\int \frac{dx}{1+\sqrt{x}}$ Let $u = 1 + \sqrt{x}$

$= \int \frac{2(u-1)du}{u}$ $\therefore du = \frac{1}{2\sqrt{x}}$

$= \int \left(2 - \frac{2}{u}\right) du$ $\therefore dx = 2\sqrt{x}du$

$= 2u - 2ln|u| + C$ $\therefore dx = 2(u - 1)du$

$= 2(1 + \sqrt{x}) - 2ln|1 + \sqrt{x}| + C$

So $\int_0^1 \frac{dx}{1+\sqrt{x}}$ = $\left[2(1+\sqrt{x}) - 2ln|1+\sqrt{x}|\right]_0^1$ = $2 + 2\sqrt{1} - 2ln|2| - 2 - 2\sqrt{0} + 2ln|1|$ = $2 - 2ln2$.

Q22: $\int (x+1)arctanxdx$ $\quad f(x) = arctanx \quad g'(x) = (x+1)$

$= \frac{(x+1)^2}{2} arctanx - \int \frac{(x+1)^2}{2} \times \frac{1}{1+x^2} dx$ $\quad f'(x) = \frac{1}{1+x^2} \quad g(x) = \frac{(x+1)^2}{2}$

$= \frac{(x+1)^2}{2} arctanx - \frac{1}{2}\int \frac{x^2+2x+1}{1+x^2} dx$

$= \frac{(x+1)^2}{2} arctanx - \frac{1}{2}\int \left(1 + \frac{2x}{1+x^2}\right) dx$ $\quad$ let $u = 1 + x^2$ $\quad \therefore du = 2xdx$

$= \frac{(x+1)^2}{2} arctanx - \frac{x}{2} - \frac{1}{2}\int \frac{1}{u} du$

$= \frac{(x+1)^2}{2} arctanx - \frac{x}{2} - \frac{1}{2}\ln(1+x^2) + C$

$= \frac{(x+1)^2}{2} arctanx - \frac{x}{2} - \ln\sqrt{1+x^2} + C$

Q23: Solve $\frac{dy}{dx} = \frac{y(x+y)}{x^2}$ for y, given $y = 1$ when $x = 1$. $\quad$ Let $v = \frac{y}{x}$

$\therefore \frac{dy}{dx} = \frac{xy+y^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 = v + v^2$ $\quad \therefore y = vx$

$\therefore v + x\frac{dv}{dx} = v + v^2$ $\quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore x\frac{dv}{dx} = v^2$

$\therefore \int \frac{dv}{v^2} = \int \frac{dx}{x}$

$\therefore -v^{-1} = ln|x| + C$

$\therefore -\frac{x}{y} = ln|x| + C$

When $x = 1, y = 1$ and so $-1 = ln|1| + C$, so $C = -1$.

$\therefore \frac{x}{y} = 1 - ln|x|$

$\therefore y = \frac{x}{1-ln|x|}$.

Q24(a): $2\frac{dv}{dt} = 2g - cv$

$\therefore \int \frac{dv}{2g-cv} = \int \frac{dt}{2}$

$\therefore \frac{ln|2g-cv|}{-c} = \frac{t}{2} + k$ (k a constant)

$\therefore ln|2g-cv| = -\frac{ct}{2} - kc$

$\therefore 2g - cv = e^{-\frac{ct}{2}-kc} = k_1 e^{-\frac{ct}{2}}$, where $k_1 = e^{-kc}$

When $t = 0, v = 0$, so $2g = k_1$.

$\therefore 2g - cv = 2ge^{-\frac{ct}{2}}$

$\therefore\ cv = 2g - 2ge^{-\frac{ct}{2}}$

$\therefore\ v = \frac{2g}{c}\left(1 - e^{-\frac{ct}{2}}\right).$

(b) As $t \to \infty$, $e^{-\frac{ct}{2}} \to 0$, so $v \to \frac{2g}{c}$. So the limiting velocity is $\frac{2g}{c}$.

SECTION E

Question 25 **(3 marks)**

Given *z* is a complex number such that $\text{Arg}(z) = \theta$ and $|z| = r$, write an expression for each of $\text{Arg}(4z^3)$ and $|4z^3|$ in terms of *r* and/or *θ*.

$\text{Arg}(4z^3) = \text{Arg}(z^3) = 3\,\text{Arg}(z) = 3\theta$;

$|4z^3| = 4|z^3| = 4 \times |z|^3 = 4r^3$.

Question 26 **(3 marks)**

Find all complex numbers *z* such that $|z - 2i| = 3$ and $\text{Im}(z) = 1$.

If we say that $z = a + bi$ then $\text{Im}(z) = 1$ gives $z = a + i$ and $z - 2i = a - i$.

The statement that $|z - 2i| = 3$ then becomes $|a - i| = 3$ gives $\sqrt{a^2 + 1} = 3$.

Squaring both sides (alert to the possibility of a spurious solution this might create) gives $a^2 + 1 = 9$, so that $a^2 = 8$.

Thus $a = \pm 2\sqrt{2}$.

This gives the two putative solutions $z = 2\sqrt{2} + i$ and $z = -2\sqrt{2} + i$.

Both solutions are admissible.

An alternative is to draw a diagram.

Question 27 **(3 marks)**

Two roots of a polynomial equation $P(z) = 0$ are $2i$ and 5. Write a suitable polynomial, *P*, of least positive degree given that its coefficients are:

(a) complex numbers;

(b) real numbers;

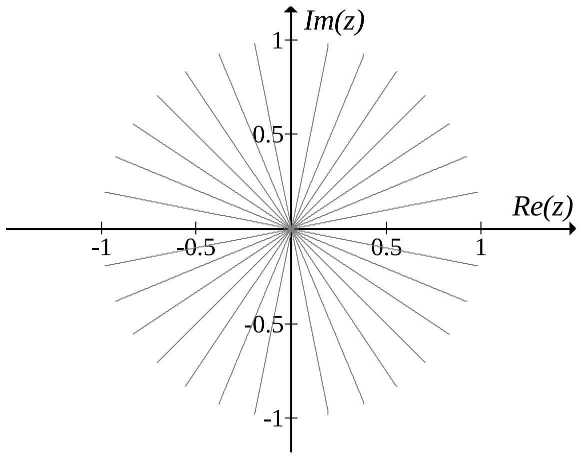
(a) $P(z) = (z-2i)(z-5) = z^2 - (5+2i)z + 10i$

(b) We employ the complex conjugate:

$$P(z) = (z-2i)(z+2i)(z-5) = z^3 - 5z^2 + 4z - 20$$

Question 28 **(6 marks)**

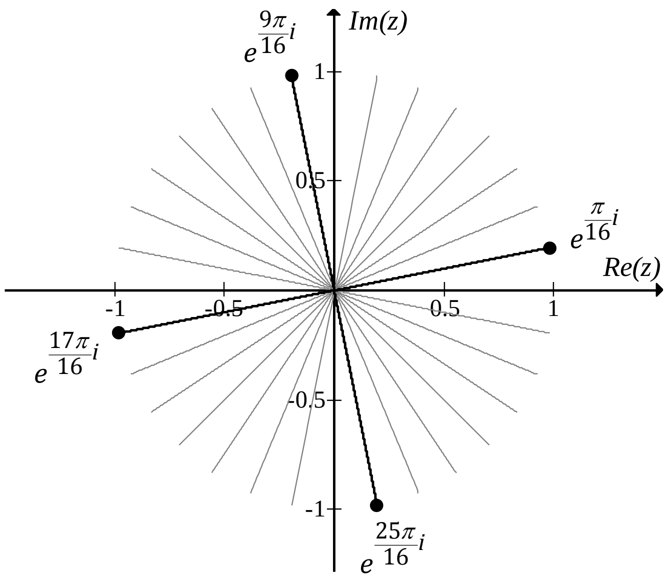
(a) Use an algebraic method to calculate at least one solution to the equation $z^4 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ in polar form, and then neatly sketch all solutions of the equation on the Argand plane. Label each solution in your sketch.



By De Moivre or otherwise the solutions are (or one solution is) obtained:

$$z^4 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \cos\left(\frac{\pi}{4} + 2n\pi\right) + i\sin\left(\frac{\pi}{4} + 2n\pi\right) \Rightarrow z = \cos\left(\frac{\pi}{16} + \frac{n\pi}{2}\right) + i\sin\left(\frac{\pi}{16} + \frac{n\pi}{2}\right).$$

So one solution is $\cos\frac{\pi}{16} + i\sin\frac{\pi}{16} = e^{\frac{\pi i}{16}}$, and all *four* solutions are at intervals of $\frac{\pi}{2}$.



(b) If n is a positive integer, prove that

$$\left(1+\sqrt{3}\,i\right)^n+\left(1-\sqrt{3}\,i\right)^n=2^{n+1}\cos\frac{n\pi}{3}.$$

We note that:

$1+\sqrt{3}\,i=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$; and $1-\sqrt{3}\,i=2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)=2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$,

so that we can use DeMoivre's Theorem:

$$\begin{aligned}\left(1+\sqrt{3}\,i\right)^n+\left(1-\sqrt{3}\,i\right)^n&=2^n\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n+2^n\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n\\&=2^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right)+2^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)\\&=2\times 2^n\cos\frac{n\pi}{3}\\&=2^{n+1}\cos\frac{n\pi}{3}\end{aligned}$$

Question 29 **(7 marks)**

Given that $2i$ is a solution of the equation $z^4-2z^3+mz^2+nz+104=0$, find the value of the real numbers m and n and find the other three solutions.

We have

$$\begin{aligned}(2i)^4-2(2i)^3+m(2i)^2+n(2i)+104&=0\\16+16i-4m+2ni+104&=0+0i\\(120-4m)+i(16+2n)&=0+0i\end{aligned}$$

Equating real and imaginary parts gives $120=4m\Rightarrow m=30$ and $16+2n=0\Rightarrow n=-8$.

So we have $z^4-2z^3+30z^2-8z+104=0$.

By the CRT, $-2i$ is also a root, so we have

$$\begin{aligned}(z-2i)(z+2i)\left(az^2+bz+c\right)&=0\\\left(z^2+4\right)\left(az^2+bz+c\right)&=0\end{aligned}$$

Simple coefficient comparisons then reveal that a = 1 and c = 26.

Also, $4bz=-8z$ gives $b=-2$.

So the second quadratic factor is $az^2+bz+c=z^2-2z+26$.

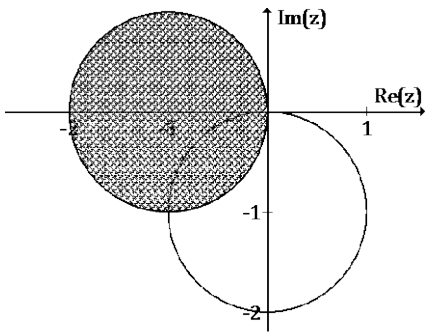
Solving $z^2 - 2z + 26 = 0$ gives $z = \frac{2 \pm \sqrt{4 - 4 \times 26}}{2} = \frac{2 \pm 2\sqrt{-25}}{2} = 1 \pm 5i$.

So we see that the solutions are $z \in \{\pm 2i, 1 \pm 5i\}$.

Question 30 **(8 marks)**

(a) On one Argand diagram illustrate the regions $|z+1| \le 1$ and $|z+i| = 1$.

For Part (a), the condition $|z+1| \le 1$ determines a disk, radius 1, centred on $-1 + 0i$, and the region $|z+i| = 1$ determines a circle circumference, centred on $0 - i$ with radius 1.



(b) Determine:

(i) the equation in Cartesian form that describes all points satisfying $|z+1| = |z+i|$.

(ii) the least value of $|z+i|$ given that $|z+1| \le 1$.

For Part (b) (i), $|z+1| = |z+i|$ specifies the perpendicular bisector of the line segment joining the centres of the circles shown above (passing through the intersections of the circles).

Since the line joining $(-1, 0)$ and $(0, -1)$ has slope –1 and midpoint $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, the line we are after has slope 1 and passes through $\left(-\frac{1}{2}, -\frac{1}{2}\right)$. Its equation is, therefore, $y = x$.

For Part (b) (ii), the expression $|z+i|$ can be interpreted as the distance of the general number *z* from $0 - i$, the centre of the circle.

The requirement that $|z+1| \le 1$ means that we require the general number *z* to be in that disk region, shown above. Specifically, since we require the least distance from $0 - i$, we require *z* to be on the circumference of the disk and on the line segment joining the centres.

The centres have separation $\sqrt{2}$ (Pythagoras) and, since the disk has radius 1, the least distance (shown in bold, right) is $\sqrt{2} - 1$ units.