OFFICE OF TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

Tasmanian Certificate of Education
External Assessment 2016

PLACE YOUR CANDIDATE LABEL HERE

MATHEMATICS SPECIALISED

(MTS415114)

Time allowed for this paper

- Working time: 3 hours
- Plus 15 minutes recommended reading time

Pages:	16
Questions:	30
Attachment:	Information Sheet

Candidate Instructions

1. You **MUST** make sure that your responses to the questions in this examination paper will show your achievement in the criteria being assessed.
2. There are **FIVE** sections to this paper.
3. You must answer **ALL** questions.
4. Answer each section in a separate answer booklet.
5. It is recommended that you spend approximately 36 minutes on each section.
6. The 2016 Information Sheets for Mathematics Specialised and Mathematics Methods can be used throughout the examination (provided with the paper).
7. No other written material is allowed into the examination.
8. All written responses must be in English.

On the basis of your performance in this examination, the examiners will provide results on each of the following criteria taken from the course statement:

Criterion 4 Demonstrate an understanding of finite and infinite sequences and series.

Criterion 5 Demonstrate an understanding of matrices and linear transformations.

Criterion 6 Use differential calculus and apply integral calculus to areas and volumes.

Criterion 7 Use techniques of integration and solve differential equations.

Criterion 8 Demonstrate an understanding of complex numbers.

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Additional Instructions for Candidates

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

You are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching and any approved scientific or graphics or CAS calculator (memory may be retained). **Unless instructed otherwise**, calculators may be used to their full capacity when undertaking this examination.

SECTION A – Sequences and Series

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a separate answer booklet for this section.

This section assesses **Criterion 4**.

Question 1 (3 marks)

Find $\sum_{r=1}^{n} 2^{-r}$.

Question 2 (3 marks)

Find a factorised expression for the sum of the first n terms of the series whose r^{th} term is $r(3r-1)$.

Question 3 (5 marks)

Prove that the sequence $\left\{\frac{n^3-n^2+n-1}{n^3-1}\right\}$ converges to 1.

Question 4 (5 marks)

Prove by mathematical induction that $2+3.2+4.2^2+5.2^3+...+(n+1)2^{n-1}=n.2^n$, for all integers $n\geq 1$.

Section A continues.

Question 5 (7 marks)

(a) Show that $\frac{1}{r^2-1}=\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$.

(b) (i) Using the result from Part (a), prove that

$$\sum_{r=2}^{n}\frac{1}{\left(r^2-1\right)}=\frac{3n^2-n-2}{4n\left(n+1\right)}.$$

(ii) State the value of $\sum_{r=2}^{\infty}\frac{1}{\left(r^2-1\right)}$.

Question 6 (7 marks)

The following is the sum of the first six terms of a series:

$$1+2^2+3+4^2+5+6^2+....$$

(a) Develop an expression ***in fully factorised form*** for the **sum of the first *n* terms** of the series, where ***n* is even**.

(b) Use your expression to find the sum of the first 97 terms.

SECTION B – Matrices and Linear Transformations

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a separate answer booklet for this section.

This section assesses **Criterion 5**.

Question 7 (3 marks)

Show that for matrices $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -3 & 3 \end{pmatrix}$, matrix multiplication is not commutative.

Question 8 (3 marks)

Find all values of *k* for which the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} - k\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ has an inverse.

Question 9 (5 marks)

A 2 × 2 matrix P is called **idempotent** if $P^2 = P$.

(a) Show that the matrix $\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$ is idempotent.

(b) Show that all **singular** 2 × 2 matrices of the form $\begin{pmatrix} \frac{1}{2} & b \\ c & \frac{1}{2} \end{pmatrix}$ are idempotent.

Section B continues.

Section B (continued)

Question 10 (6 marks)

(a) Recalling $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, show that the matrix of a reflection in the line $y = \frac{1}{2}x$ is $\frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

(b) Show that under reflection in the line $y = \frac{1}{2}x$, the *midpoint* of the line segment joining $(10, 10)$ to $(20, 0)$ is mapped to the *midpoint of the images* of $(10, 10)$ and $(20, 0)$.

Question 11 (6 marks)

The 3 × 3 system of equations

$$\begin{aligned} 6x + 2y - 10z &= 0 \\ 4y - 8z &= 12 \\ 14x + 7y - 12z &= k \end{aligned}$$

where $k \in Z^+$, describes three planes that intersect in a single point.

(a) Use Gauss-Jordan elimination to show that the system reduces to the augmented matrix with form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \dfrac{k-23}{16} \\ 0 & 1 & 0 & \dfrac{17+k}{8} \\ 0 & 0 & 1 & \dfrac{k-7}{16} \end{array}\right)$$

(b) Determine the least positive integral value of k such that the three coordinates of the intersection point are integers. State the point of intersection.

Question 12 (7 marks)

An ellipse is subjected to a reflection in the line $y = x$, followed by an anticlockwise rotation through $\frac{\pi}{4}$ radians.

(a) Determine the matrix that represents the composition of these two transformations.

(b) The image of the source ellipse under the composition of these two transformations is itself an ellipse with equation $\dfrac{x^2}{9} + y^2 = 1$.

Determine the equation of the source ellipse in a form with integer coefficients.

(c) This composition is equivalent to a reflection in the line $y = x\tan(\varphi)$. Determine a value of φ.

SECTION C – Differential Calculus, Areas and Volumes

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a separate answer booklet for this section.

This section assesses **Criterion 6**.

Question 13 (3 marks)

Determine the Maclaurin series expansion for $\frac{1}{1-x}$ as far as the term in x^3.

Question 14 (3 marks)

If y = arctan x, find the value of a given that $\frac{d^2y}{dx^2} = ax\left(\frac{dy}{dx}\right)^2$, for all x.

Question 15 (5 marks)

A curve is defined implicitly by $\sqrt{x} + \sqrt{y} = 1$.

Find the equation of the tangent to the curve at the point with $x = \frac{1}{4}$.

Question 16 (5 marks)

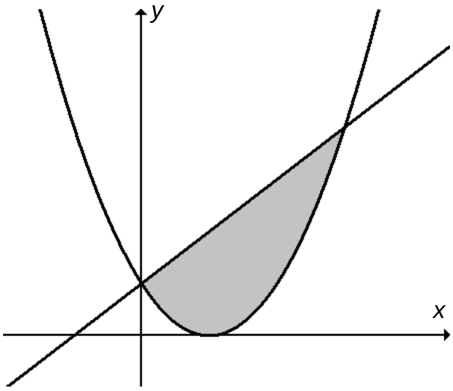
A curve is given by the equation $y = x\sqrt{2-x}$ for x £ 2.

Using a suitable substitution, evaluate an integral and hence find the exact area of the region enclosed by the curve and the x-axis between its zeros.

Section C continues.

Question 17 (7 marks)

The diagram shows the region enclosed by the curves $y = x + 1$ and $y = (x - 1)^2$. The region is rotated about the y-axis.



Find the exact volume of the solid formed.

(You may use your calculator to evaluate integrals, but you must make it clear how you have set up any integrals to calculate the volume.)

Question 18 (7 marks)

Let $f(x) = e^x - e^{1-x} - (e+1)x$.

(a) Find and determine the nature of any stationary points and points of inflection on the curve $y = f(x)$.

(b) Sketch the curve. You are not required to calculate any zeros algebraically.

SECTION D – Integral Calculus

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a separate answer booklet for this section.

This section assesses **Criterion 7**.

Question 19 (3 marks)

Without using your calculator, find the exact value of $\int_0^{\frac{\pi}{3}} \tan x dx$.

Question 20 (3 marks)

By completing the square, find $\int \frac{1}{x^2+4x+5} dx$.

Question 21 (5 marks)

Using a suitable substitution, find the exact value of $\int_0^1 \frac{dx}{1+\sqrt{x}}$.

Question 22 (5 marks)

Using integration by parts, find $\int (x+1) \arctan x \, dx$.

Question 23 (7 marks)

Solve the differential equation $\frac{dy}{dx} = \frac{y(x+y)}{x^2}$ for y, given that y = 1 when x = 1.

Section D continues.

Question 24 (7 marks)

An object, initially at rest, is dropped from a tall building. The speed, v, of the object at time t, satisfies the differential equation

$$2\frac{dv}{dt} = 2g - cv,$$

where g and c are constants.

(a) Solve the differential equation to show that $v = \frac{2g}{c}\left(1 - e^{\frac{-ct}{2}}\right)$.

(b) What is the limiting value of v?

SECTION E – Complex Numbers

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a separate answer booklet for this section.

This section assesses **Criterion 8**.

Question 25 (3 marks)

Given z is a complex number such that $\text{Arg}(z) = q$ and $|z| = r$, write an expression for each of $\text{Arg}(4z^3)$ and $|4z^3|$ in terms of r and/or θ.

Question 26 (3 marks)

Find all complex numbers z such that $|z - 2i| = 3$ and $\text{Im}(z) = 1$.

Question 27 (3 marks)

Two roots of a polynomial equation $P(z) = 0$ are $2i$ and 5. Write a suitable polynomial, P, of least positive degree given that its coefficients are:

(a) complex numbers;

(b) real numbers.

Section E continues.

Section E (continued)

Question 28 (6 marks)

(a) Use an algebraic method to calculate at least one solution to the equation $z^4 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ in polar form, and then neatly sketch all solutions of the equation on an Argand plane. Label each solution in your sketch.

(b) If n is a positive integer, prove that

$$\left(1+\sqrt{3}\,i\right)^n + \left(1-\sqrt{3}\,i\right)^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right).$$

Question 29 (7 marks)

Given that $2i$ is a solution of the equation $z^4 - 2z^3 + mz^2 + nz + 104 = 0$, find the value of the real numbers m and n and find the other three solutions.

Question 30 (8 marks)

(a) On an Argand diagram illustrate the regions $|z+1| \le 1$ and $|z+i| = 1$.

(b) Determine:

(i) the equation in Cartesian form that describes all points satisfying $|z+1| = |z+i|$;

(ii) the least value of $|z+i|$ given that $|z+1| \le 1$.

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