OFFICE OF TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

Tasmanian Certificate of Education
External Assessment 2017

PLACE YOUR CANDIDATE LABEL HERE

MATHEMATICS SPECIALISED

(MTS415114)

Time allowed for this paper

- Working time: 3 hours
- Plus 15 minutes recommended reading time

Pages:	16
Questions:	30
Attachments:	2 Information Sheets

Candidate Instructions

1. You **MUST** make sure that your responses to the questions in this examination paper will show your achievement in the criteria being assessed.
2. There are **FIVE** sections to this paper.
3. You must answer **ALL** questions.
4. Answer each section in a **SEPARATE** answer booklet.
5. It is recommended that you spend approximately 36 minutes on each section.
6. The 2017 Information Sheets for Mathematics Specialised and Mathematics Methods can be used throughout the examination (provided with the paper).
7. No other written material is allowed into the examination.
8. All written responses must be in English.

On the basis of your performance in this examination, the examiners will provide results on each of the following criteria taken from the course statement:

Criterion 4 Demonstrate an understanding of finite and infinite sequences and series.

Criterion 5 Demonstrate an understanding of matrices and linear transformations.

Criterion 6 Use differential calculus and apply integral calculus to areas and volumes.

Criterion 7 Use techniques of integration and solve differential equations.

Criterion 8 Demonstrate an understanding of complex numbers.

BLANK PAGE

Additional Instructions for Candidates

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

You are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching and any approved scientific or graphics or CAS calculator (memory may be retained). **Unless instructed otherwise**, calculators may be used to their full capacity when undertaking this examination.

SECTION A – Sequences and Series

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 4**.

Question 1 (3 marks)

Use a formal definition to prove that the sequence $\left\{\frac{3n-2}{5}\right\}$ diverges to infinity.

Question 2 (3 marks)

For what values of x does the sequence $\left\{(1-\ln x)^n\right\}$ converge?

Question 3 (5 marks)

Use mathematical induction to prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for all positive integers n.

Question 4

(a) Show that $\sum_{r=1}^{n} (2r-1)^2 = \frac{n(2n-1)(2n+1)}{3}$. (4 marks)

(b) Hence find the sum of the squares of the odd numbers between 100 and 200. (2 marks)

Section A continues.

Question 5

(a) Use a formal definition to prove that the sequence $\left\{\frac{n+3}{n^2-11}\right\}$ converges to zero. (4 marks)

(b) Determine the least integral value of N such that $\frac{n+3}{n^2-11}<10^{-2}$ provided $n>N$. (2 marks)

Question 6

(a) Find a factorised expression in terms of n for $\sum_{r=1}^{n}\frac{2}{r(r+1)(r+2)}$. (5 marks)

(b) Deduce that $\sum_{r=n+1}^{\infty}\frac{2}{r(r+1)(r+2)}=\frac{1}{(n+1)(n+2)}$. (2 marks)

SECTION B – Matrices and Linear Transformations

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 5**.

Question 7 (3 marks)

Matrices $\mathbf{A}$ and $\mathbf{I}$ are given by $\mathbf{A}=\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$ and $\mathbf{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Show that $\mathbf{A}^2=4\mathbf{A}-\mathbf{I}$.

Question 8

(a) Given non-singular matrices $\mathbf{X}$ and $\mathbf{Y}$, simplify $\mathbf{XY}\left(\mathbf{X}^{-1}\mathbf{Y}\right)^{-1}$. (2 marks)

(b) If $\mathbf{Z}$ is the matrix obtained by multiplying the first row of a 2×2 matrix $\mathbf{M}$ by the real number k, show that $\det(\mathbf{Z})=k\det(\mathbf{M})$. (2 marks)

Question 9

Given the matrix $\mathbf{A}=\begin{pmatrix}k & 1\\ 1 & 4k\end{pmatrix}$:

(a) Find all values of real k such that the matrix $\mathbf{A}$ is non-singular. (2 marks)

(b) Interpret geometrically the linear transformation represented by the matrix $\mathbf{A}$ when $k=0$. (2 marks)

Section B continues.

Section B (continued)

Question 10 (5 marks)

The matrix $\mathbf{Z}$ is given by $\mathbf{Z}=\begin{pmatrix} \frac{1}{2} & \frac{2+\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1-2\sqrt{3}}{2} \end{pmatrix}$.

The transformation represented by $\mathbf{Z}$ is equivalent to the transformation represented by $\mathbf{X}=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ followed by another transformation represented by the matrix $\mathbf{Y}$.

Find $\mathbf{Y}$ and describe fully its geometrical effect.

Question 11

Consider the system of equations

$$\begin{aligned} x-y+z&=a \\ x+y+9z&=b \\ 2x-y+6z&=c \end{aligned}$$

in which a, b and c are real numbers.

(a) Show that $3a+b-2c=0$ is required for the system of equations to have a solution in x, y and z.

Note: you may use your calculator to perform any necessary row reductions, but you must explain your steps clearly. (4 marks)

(b) If $a=1$, $b=5$ and $c=4$, state the solution to the system of equations and provide a geometrical interpretation. (3 marks)

Question 12

The graph with equation $y=\frac{1}{x}$ undergoes an anticlockwise rotation through $\frac{\pi}{3}$ followed by a reflection in the line $y=\frac{x}{\sqrt{3}}$ and then a dilation in the x-axis (parallel to the x-axis), factor 2.

(a) Find the equation of the image of the graph. (6 marks)

(b) The circle with centre $(2,2)$ and radius 3 undergoes the same sequence of transformations. Determine the area of the image of the circle. (1 mark)

SECTION C – Differential Calculus, Areas and Volumes

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 6**.

Question 13 (3 marks)

Sketch a graph $y = f(x)$ with the following properties:

$$f(x)\begin{cases} >0 & x<0,\ x>4 \\ =0 & x=0,\ x=4 \\ <0 & 0<x<4 \end{cases} \qquad f'(x)\begin{cases} >0 & 1<x<4,\ x>4 \\ =0 & x=1,\ x=4 \\ <0 & x<1 \end{cases} \qquad f''(x)\begin{cases} >0 & x<2,\ x>4 \\ =0 & x=2,\ x=4 \\ <0 & 2<x<4 \end{cases}$$

Question 14 (3 marks)

Use the trapezoidal rule with four intervals to determine the approximate area under the curve with equation $y = \arcsin x$ for $0 < x < 1$. Give the area correct to two decimal places.

Question 15

For the curve with equation $x^3 + y^3 = 4xy$

(a) Find $\dfrac{dy}{dx}$. (3 marks)

(b) Find the equation of the tangent to the curve at the point $(2, 2)$. (2 marks)

Section C continues.

Section C (continued)

Question 16

(a) Show that the equation of the normal to the hyperbola $xy=1$ at the point $P\left(p,\frac{1}{p}\right)$ is given by

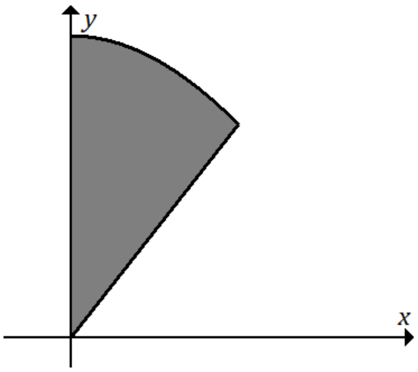
$$y-\frac{1}{p}=p^2(x-p)\ .$$ (3 marks)

(b) Prove that this normal meets the hyperbola again at the point $Q\left(q,\frac{1}{q}\right)$ where $q=-\frac{1}{p^3}$.

(2 marks)

Question 17

Parts of the graphs with equations $y=\sqrt{2}\cos x$ and $y=\frac{4x}{\pi}$ are shown. The area bounded by the curves and the y-axis is shaded.



(a) Show that these curves intersect at $\left(\frac{\pi}{4},1\right)$. (1 mark)

(c) Determine the exact value of the shaded area. (2 marks)

(d) Determine the exact volume when the shaded area is rotated about the x-axis. (3 marks)

Question 18

Consider the function $f(x)=(x+1)e^{x-1}$.

(a) Show that $x=-1$ is the only zero. (1 mark)

(b) Show that $\left(-2,-\frac{1}{e^3}\right)$ is the only stationary point and that it is a global minimum. (3 marks)

(c) Find and classify all points of inflection. (2 marks)

(d) Sketch the graph of the function $f(x)=(x+1)e^{x-1}$. (2 marks)

SECTION D – Integral Calculus

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 7**.

Question 19 (3 marks)

By using an appropriate trigonometric identity obtain the exact value of $\int_0^1 \tan^2\left(\frac{\pi x}{4}\right) dx$.

Question 20 (4 marks)

Use partial fractions to determine $\int \frac{x+3}{x(x-1)} dx$.

Question 21 (4 marks)

Determine $\int \sin^3 x \, dx$.

Question 22

Consider the differential equation $x\frac{dy}{dx} = x^2 y - y$ for $x > 0$ and $y > 0$.

(a) Solve the differential equation. (3 marks)

(b) Given that $y = 1$ when $x = 1$, find y explicitly in terms of x. (2 marks)

Section D continues.

Section D (continued)

Question 23

Let $I_n = \int_0^1 x^n e^x \, dx$, where $n \geq 0$.

(a) Find I_1 . (3 marks)

(b) Show that $I_n = e - nI_{n-1}$ for all $n \geq 1$, and hence find I_3 . (4 marks)

Question 24 (7 marks)

Solve the differential equation $3xy^2 \frac{dy}{dx} = x^3 + 4y^3$ given that $y = 0$ when $x = 1$.
Express your answer in the explicit form $y = f(x)$.

SECTION E – Complex Numbers

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 8**.

Question 25 (3 marks)

If the principal argument of the complex number $1+ai$ is $-\frac{\pi}{6}$, find the value of the real number a.

Question 26 (3 marks)

Describe the geometrical effect in the Argand plane of multiplying a complex number by $\frac{1-i}{1+i}$.

Question 27 (6 marks)

An equilateral triangle in the complex plane has one vertex at the origin $(0, 0)$. Another vertex represents the complex number $7-\sqrt{3}i$. Determine the complex numbers represented by the two possible locations of the third vertex.

Provide a diagram to illustrate your solution.

Section E continues.

Section E (continued)

Question 28 (6 marks)

On an Argand diagram shade the intersection of the two regions defined by

$$\{z : |z-1+i| \geq |z+1-i|\} \quad \text{and} \quad \{z : |z-1-2i| \leq 1\}.$$

Show all key points on your diagram.

Question 29

(a) Solve $z^5 + 1 = 0$. Present your solutions in polar form. (3 marks)

(b) Hence solve $z^9 - 16z^5 + z^4 - 16 = 0$. (3 marks)

Question 30

(a) Use an algebraic method to determine the real numbers a and b if $z - 7i$ is a factor of $p(z) = z^4 + az^3 + bz^2 - 98z + 98$. (3 marks)

(b) Hence solve $p(z) = 0$. (3 marks)

BLANK PAGE

BLANK PAGE

ATSC

OFFICE OF TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION