OFFICE OF TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

Tasmanian Certificate of Education
External Assessment 2018

PLACE YOUR CANDIDATE LABEL HERE

MATHEMATICS SPECIALISED
(MTS415118)

Time allowed for this paper

- Working time: 3 hours
- Plus 15 minutes recommended reading time

Pages:	16
Questions:	30
Attachments:	2 Information Sheets

Candidate Instructions

1. You **MUST** make sure that your responses to the questions in this examination paper will show your achievement in the criteria being assessed.
2. There are **FIVE** sections to this paper.
3. You must answer **ALL** questions.
4. Answer each section in a **SEPARATE** answer booklet.
5. It is recommended that you spend approximately 36 minutes on each section.
6. The Information Sheets for Mathematics Specialised and Mathematics Methods can be used throughout the examination (provided with the paper).
7. No other written material is allowed into the examination.
8. All written responses must be in English.

On the basis of your performance in this examination, the examiners will provide results on each of the following criteria taken from the course statement:

Criterion 4 Solve problems and use techniques involving finite and infinite sequences and series.

Criterion 5 Solve problems and use techniques involving matrices and linear algebra.

Criterion 6 Use differential calculus and apply integral calculus to areas and volumes.

Criterion 7 Use techniques of integration and solve differential equations.

Criterion 8 Solve problems and use techniques involving complex numbers.

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Additional Instructions for Candidates

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

You are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching and any approved scientific or graphics or CAS calculator (memory may be retained). **Unless instructed otherwise**, calculators may be used to their full capacity when undertaking this examination.

SECTION A

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 4**.

Question 1 (3 marks)

A sequence $\{u_n\}$ is defined by: $u_1 = 2$, $u_{n+1} = \frac{1}{u_n}$ for $n \geq 1$.

Find u_{100} .

Question 2 (4 marks)

If the sum to n terms of a series is given by $S_n = n(n+2)$, find the nth term, u_n .

Question 3

(a) Find the first four terms in the Maclaurin expansion of $f(x) = \frac{1}{1+2x}$. (4 marks)

(a) Hence, find the first four terms in the Maclaurin expansion of $\ln(1+2x)$. (2 marks)

Question 4 (6 marks)

Use a formal proof to show that $\lim_{n \to \infty} \frac{n^2}{1+n+n^2} = 1$.

Section A continues.

Section A (continued)

Question 5

(a) Show that $\frac{1}{2r-1}-\frac{1}{2r+1}=\frac{2}{4r^2-1}$ for any integer r. (2 marks)

(b) Using this result, show that $\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{n}{2n+1}$. (4 marks)

(c) Hence, find $\sum_{r=100}^{\infty}\frac{1}{4r^2-1}$. (3 marks)

Question 6

The table below provides some expressions and calculations for the series:
$S_n=n^2-(n-1)^2+(n-2)^2-(n-3)^2+.....+(-1)^{n-1}\ \ n\in Z^+$

n	S_n	
1	1^2	1
2	2^2-1^2	3
3	$3^2-2^2+1^2$	6
⋮	⋮	⋮
n	$n^2-(n-1)^2+(n-2)^2-(n-3)^2+.....+(-1)^{n-1}$	

(a) Evaluate S_4. (1 mark)

(b) Write down an expression for S_{n+1} and hence show that $S_{n+1}=(n+1)^2-S_n$. (3 marks)

(c) Prove using mathematical induction that
$n^2-(n-1)^2+(n-2)^2-(n-3)^2+.....+(-1)^{n-1}=\frac{n(n+1)}{2}\ \ n\in Z^+$. (4 marks)

SECTION B

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 5**.

Question 7

Given matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 8 \end{pmatrix}$

Find:

(a) $\mathbf{A}^2$. (2 marks)

(b) $\mathbf{A}^{-1}$. (2 marks)

Question 8

The transpose of a matrix is obtained by writing rows as columns. In the case of a 2×2 matrix

$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the transpose is written $\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) Show that $|\mathbf{A}| = |\mathbf{A}^{\mathrm{T}}|$ for any 2×2 matrix $\mathbf{A}$. (2 marks)

(b) If $\mathbf{R}$ is the matrix of a rotation through angle θ, show that $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. (2 marks)

Section B continues.

Section B (continued)

Question 9

After reflection in the line $\boldsymbol{y}=\dfrac{1}{\sqrt{3}}\boldsymbol{x}$, the equation of the image of a line is given by $x\sqrt{3}-y=2$.

(a) Find the equation of the original line. (4 marks)

(b) On a diagram, sketch and label the original, image and reflection lines showing any key points. (2 marks)

Question 10

Consider the following system of equations for x, y and z

$$\begin{aligned} x-2y+2z&=1\\ -x+3y-z&=k\\ 2x-5y+kz&=-2 \end{aligned}$$

where k is a real number.

(a) Set up an augmented matrix for the system of equations and use the Gauss-Jordan method to transform it to reduced row echelon form, when $k\neq 3$. (3 marks)

(b) Hence solve the system, in terms of k, when $k\neq 3$. (1 mark)

(c) If $k=3$, given $\boldsymbol{z}=\boldsymbol{t}$, find x and y in terms of t and give a geometrical interpretation. (3 marks)

Question 11

(a) Find the equation of the plane containing the points (1,4,2), (3,2,1) and (5,3,5). (4 marks)

You may use your calculator to carry out any matrix manipulation.

(b) Show that the line given by: $x=1+2t; y=1+t; z=-3+4t$ lies in the plane. (3 marks)

Question 12

(a) Draw on a diagram the unit square OABC (with vertices O(0,0), A(1,0), B(1,1) and C(0,1)) and its image under the transformation given by the matrix $\mathbf{M}=\begin{pmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{pmatrix}$. (2 marks)

(b) Given that $\mathbf{M}$ is a combination of dilations and a rotation, find and describe these fully. (3 marks)

(c) Find and describe the image of the circle $\boldsymbol{x}^2+\boldsymbol{y}^2=\mathbf{1}$ under the transformation given by the matrix $\mathbf{K}=\begin{pmatrix}k & 1\\ -1 & k\end{pmatrix}$ where k is any real number. (3 marks)

SECTION C

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 6**.

Question 13 (4 marks)

Show that $y = \arctan x$ is a solution of the equation $\left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$.

Question 14 (4 marks)

Show that if $x^2y^2 = 1$ then $\frac{dy}{dx} = -\frac{y}{x}$ and $\frac{d^2y}{dx^2} = \frac{2y}{x^2}$.

Question 15

(a) Show that the points $(-1,0)$ and $\left(0,\frac{\pi}{2}\right)$ both lie on the graph of the curve given by $\cos y + y\sin x = x^2$. (2 marks)

(b) Show that the tangent to the curve at $\left(0,\frac{\pi}{2}\right)$ also passes through the point $(-1,0)$. (4 marks)

Section C continues.

Section C (continued)

Question 16

(a) Show that $\frac{d}{dx}(2^{\sqrt{1-x}}) = \frac{-(\ln 2)2^{\sqrt{1-x}}}{2\sqrt{1-x}}$. (3 marks)

(b) Hence, find the area bound by the graph of the function $y = \frac{2^{\sqrt{1-x}}}{\sqrt{1-x}}$ and the x-axis on the interval $-3 \le x \le 0$. (4 marks)

Question 17

Consider the function $f(x) = x(\ln x)^2$ defined for $x > 0$.

(a) Find and classify all stationary points and points of inflection for the curve $y = f(x)$. (6 marks)

(b) Sketch the curve $y = f(x)$ showing all significant points. (2 marks)

Question 18

(a) Sketch a graph of the curve $y = \arcsin(x)$ $0 \le x \le 1$. (1 mark)

(b) Find the area bounded by the curve, the x-axis, and the line $x = 1$. (3 marks)

(c) Find the volume generated when this area is rotated around the y-axis. (3 marks)

SECTION D

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 7**.

Question 19 (4 marks)

Find $\int \frac{x+3}{x(x-1)}\,dx$.

Question 20 (4 marks)

Find $\int_1^e x^3(\ln x)\,dx$.

Question 21

(a) Given $y = x\sqrt{1-x^2} - \arccos x$, find $\frac{dy}{dx}$ in simplified form. (3 marks)

(b) Using the result from part (a), find $\int_0^1 \sqrt{1-x^2}\,dx$. (3 marks)

Section D continues.

Section D (continued)

Question 22

A model for the area A of an oil spill satisfies the differential equation

$$\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{t^2}$$

where A is in square kilometres, and t is the time after the initial spill in days.

When $t = 1$, $A = 1$.

(a) Find an expression for A in terms of t. (4 marks)

(b) Describe what happens to the area in the long term. (2 marks)

Question 23

(a) Using the substitution $x = \tan u$, show that $\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)} dx = \int_a^b \frac{du}{\tan u}$

stating the values of a and b. (4 marks)

(b) Hence show that (4 marks)

$$\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)} dx = \frac{1}{2} \ln\left(\frac{3}{2}\right).$$

Question 24

Solve the differential equation (8 marks)

$$xy\frac{dy}{dx} = x^2 + 2y^2$$

for $x \geq 1$ given that $y = 0$ when $x = 1$.

Give any solutions for y explicitly in terms of x.

SECTION E

Answer **ALL** questions in this section.

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 8**.

Question 25

Let $z = 1 - i$.

(a) Show that z^2 is purely imaginary. (2 marks)

(b) Find a positive integer n such that z^n is real. (2 marks)

Question 26 (4 marks)

Given the non-zero complex number $z = a + bi$, with $a, b \in R$, prove that $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

Question 27

(a) Solve the equation $z^2 - 6z + 36 = 0$, giving your answers in the form $r cis\,\theta$. (4 marks)

(b) Given that Z is either of the solutions in part (a), deduce the exact value of Z^{-3}. (2 marks)

Section E continues.

Question 28 (6 marks)

Sketch and clearly identify the region of the Argand plane that represents the set of complex numbers

$$\left\{ z : \frac{\pi}{4} \le \operatorname{Arg}(z) \le \frac{\pi}{3} \right\} \cap \left\{ z : |z-1| \le 1 \right\}.$$

You should determine and show any key points.

Question 29

(a) Use an algebraic method to determine all real values of k for which $z-ki$ is a factor of

$P(z) = z^4 - 2z^3 + 51z^2 - 98z + 98$. (5 marks)

(b) Hence solve $P(z)=0$, giving answers in Cartesian (rectangular) form. (3 marks)

Question 30

(a) Show that for any angle θ, $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$. (3 marks)

(b) Using this result, show that $\cos^4\theta = \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$. (5 marks)

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