MATHEMATICS – SPECIALISED (MTS415118)

FEEDBACK FOR STUDENTS AND TEACHERS

GENERAL COMMENTS

Too many students used their calculators to solve problems, forgetting the description of the criteria on the exam front cover… "The assessment for *Mathematics Specialised* Level 4 will be based on the degree to which the learner can… solve problems and/or use techniques involving"…the different skills learned in this course. Also, forgetting the instruction in the 'Additional Instructions for Candidates' and at the top of each section … "You must show the method you used to solve a question. If you only show your answers you will get few, if any, marks."

SECTION A – SEQUENCES AND SERIES

QUESTION 1

No meaningful working required - students who tried to provide an explanation often made errors dealing with the exponents oscillating between 1 and -1. Many students had an incorrect understanding of odd and even.

QUESTION 2

Well done by most students. A lot of small algebraic errors in such a simple question.

QUESTION 3

(a) Most students did this perfectly

(b) Very few students saw the connection $\ln(1+2x) = 2\int f(x)\,dx$

QUESTION 4

This question was reasonably well done as a formal proof. A few students established the limit and did not prove L=1. However, working with the absolute value tripped quite a few students $|-(1+n)| = 1+n$

QUESTION 5

(a) This was done very well with the vast majority of students gaining full marks.

(b) Most students recognised the connection to part (a) but many missed the factor of a $\frac{1}{2}$

$$\sum_{r=1}^{n}\frac{1}{4r^2-1} = \frac{1}{2}\sum_{r=1}^{n}\frac{2}{4r^2-1} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right) = \cdots$$

which meant they were not able to show the required result

(c) Done quite well. A good number of students recognised that they had to split the sum into two different parts. Common errors were subtracting the sum to 100 terms (instead of 99 terms) and not being able to show that the sum to infinity was $\frac{1}{2}$

QUESTION 6

There was obvious confusion with the setting out of this question.

In fact, $S_1 = 1^2 - 0^2 + (-1)^2 - (-2)^2 + \cdots + (-1)^{1-1} \neq 1$, and similarly $S_2 \neq 3$, etc

if $n = 1, 2, \ldots$ are substituted directly into the given S_n formula.

(a) This part was mostly well done by all students.

(b) This was poorly done. Many students unsuccessfully attempted to find S_{n+1}, forgetting to give $\ldots + (-1)^{n+1-1} = \ldots + (-1)^n$. Showing RHS = … = LHS was a much more successful strategy.

Also, almost every student who attempted this did not show that …

$$-(-1)^{n-1} = \cdots + (-1)^n$$

(c) This question was reasonably well done although many students could not do the necessary algebraic working for the P_{k+1} step. The importance of the structure of a proof should be emphasised to students.

Common (and disappointing) errors were in the P_{k+1} step, for example,

$$LHS = (k+1)^2 - k^2 + (k-1)^2 - (k-2)^2 + \cdots + (-1)^k = (k+1)^2 - (-1)^k$$

SECTION B – MATRICES AND LINEAR TRANSFORMATIONS

GENERAL COMMENTS

Overall most students did well in this section, this may have been partly due to two very easy questions. The question with the new 3D content had a lot of students getting no marks.

QUESTION 7

This was a very easy question done well by all.

QUESTION 8

This was a very easy question done well by all.

QUESTION 9

This was a straight forward question done well by most. The most common error in part (a) was an algebraic mistake expanding a bracket and getting an incorrect minus sign.

Part b) drawing the graphs, many did not draw all the required lines.

QUESTION 10

Most knew what to do with this question but many made mistakes in the calculations involved in the Gauss Jordan method. To get full marks in part a) it was required to reduce the matrix to the simplest final stage

$$\begin{pmatrix} 1 & 0 & 0 & 2k-1 \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Part c) had many making errors. Just substituting $k = 3$ into the final matrix from part a) did not give the correct solution. The correct method was to go back to the line assuming $k \neq 3$ then substitute $k = 3$ and proceed from there.

QUESTION 11

This question was either done very well or very poorly. Almost a third of students received no marks for this question. For those that were familiar with this type of question it was straight forward to get full marks. A few students used a vector approach to this question, though most used matrices only. A common error was to find the equation of a line between two of the points in symmetric form rather than find the equation of the plane.

QUESTION 12

(a) Done very well by most.

(b) The results were mixed, with a common error being to look at the given matrix and assume incorrectly that there was a dilation of $\sqrt{3}$ both horizontally and vertically.

(c) This had many making a start but very few able to complete the algebra correctly to get to the final solution.

SECTION C – DIFFERENTIAL CALCULUS + AREAS & VOLUMES

In general, it was obvious that this section would take a strong student a little longer than 36 minutes to complete. The nature of this section this year was one whereby candidates were expected to be able show how results were arrived at. This meant that setting out played a more prominent part than usual for this section. There were three questions that expected candidates to show that *LHS = RHS*. There were also three instances where candidates were expected to verify that a point lay on a curve. These questions are also effectively required an *LHS = RHS* treatment.

Added to this was a Fundamental Theorem of Calculus question, which also requires a certain treatment to establish the required result. Finally, there was a question which required candidates to establish and classify stationary points and a point of inflection. In all of these there are certain requirements in terms of setting out.

QUESTION 13

The major issue here was the nature of the *LHS = RHS* "proof" required.

QUESTION 14

This question was reasonably well done. Again, appropriate setting out was important here.

QUESTION 15

Both parts of this question were reasonably well done, however, a number of candidates just substituted the co-ordinates into the equation and seemed to feel that this was all that was required. The implicit differentiation in part a) was mostly very well done and candidates seemed to be well prepared for this question.

QUESTION 16

(a) Mostly very well done in a fairly efficient manner.

(b) This was a Fundamental Theorem of Calculus question, with the word "hence" being most important. Candidates were expected to treat it as such. There were a number of ways that this could be shown. A number of students lost marks in this section for failing to establish the link between the result in part a) and the required integral in part b). Many candidates appeared to use their calculators. Some tried to integrate using parts or substitution. These candidates lost a mark if they were successful, as the instruction was "hence".

QUESTION 17

Most candidates made reasonable progress in this question. The requirement to locate and establish the turning points was well done. The requirement to verify that there was a point of inflection was where a number of candidates lost some marks. It is important to establish that there was a change of concavity by showing and noting that there was a change of sign of the second derivative and that the first derivative at the required point did not equal zero (*i.e.* there was a non-stationary point of inflexion). Few candidates showed that although the function was undefined when $x = 0$ the limit of $f(x)$ did exist at $x = 0$.

QUESTION 18

(a) Many candidates lost the mark in part a) for failing to show the points of interest. In some cases graphs were drawn without any scale on either axis.

(b) This was reasonably well done by a number of candidates. There was an expectation that students needed to use an integration by parts and then integrate by substitution to score full marks.

(c) Many candidates set up the wrong integral. Part credit was given for the candidates showing appropriate working from this point.

SECTION D – INTEGRAL CALCULUS

Overall the performance on this section of the paper was OK with the normal mix of expected behaviours. One common behaviour was to use the calculator to get over difficult parts of a problem - usually to do the calculus which rather defeats the purpose of the question. So, if there was one message to be conveyed; the questions in this section are meant to test whether the students can integrate but many seemed to think it was OK if they could demonstrate that their calculator can integrate.

QUESTION 19

Most students knew to take partial fractions and could do so accurately. Several candidates omitted absolute value signs on the log functions and/or the constant. In the main this question was not badly answered.

QUESTION 20

Most students recognised that integration by parts was required. The accuracy of the calculation was mixed; several wrote that the integral of $\ln x$ is $\frac{1}{x}$. A few made the substitution $u = \ln x$ to obtain an integral in terms of u and e^u but then did integration by parts properly.

QUESTION 21

This question was rather poorly answered. Many students did not sort out the first part correctly with the result that the reduction did not occur. Some students seemed to get the answer with little or no working; this suggested the calculator was used. The difficulty with part (a) meant that part (b) proved to be a mystery. The impression given was that several candidates realised that the integral in (b) should be related to the answer to (a) but as they had got (a) wrong there was no obvious connection. They were then stumped. Quite a number of candidates evaluated the integral by plugging into the calculator - that was easy to spot as the calculator result comes up in terms of inverse sin rather than inverse cos.

QUESTION 22

Most students knew that variables needed to be separated. Integrating $A^{-\frac{3}{2}}$ led to all sorts of problems. This question uncovered all sorts of algebraic difficulties when given an expression of the type $A^{-\frac{1}{2}} = \frac{1}{t} + c$, trying to solve for A proved beyond most. The standard 'trick' was to change the equation to $A^{\frac{1}{2}} = t + \frac{1}{c}$ and then squaring the result. This was not well answered.

QUESTION 23

Most candidates did not get this question at all. There was little understanding as to how the integral in terms of x would transform to that in terms of u with all sorts of fudges going on. Many had no idea how to find a and b. Interestingly though, many persevered with doing part (b) and did realise they needed the sin of $\frac{\pi}{3}$ and $\frac{\pi}{4}$. Apparently the importance of these values did not dawn on them for part (a). There was lots of fiddling with the logarithms that wasn't exactly convincing. Some did solve (b) properly using partial fractions. However many candidates just evaluated the integral by putting the integrand into the calculator.

QUESTION 24

This question was well done. Almost all students knew the correct substitution and obtained the appropriate reduced equation. Some of the separation of variables was suspect with the integrand being the wrong way up. Others integrated correctly but seemed to get the calculator to do it. Some got confused with evaluating the constant and then isolating 'y'; many students forgot the possibility of having the negative square root.

SECTION E – COMPLEX NUMBERS

There was evidence that some students ran out of time during this section. This would be mostly due to the previous 2 sections each being a little too long.

QUESTION 25

(a) This question was a nice easy start and the vast majority of students scored full marks for this part.

(b) This question was also very well done, however, some students did not state their answer as $n = 4$ (or similar), leaving the answer as $(1 - i)^4 = -4$

QUESTION 26

The question asked students to "prove that …". The number of students who did not use any reasonably correct setting out for a proof was disappointing. Many students assumed the result and manipulated both sides of the equation to reach their version of an answer.

QUESTION 27

(a) Those students who used the General Quadratic Formula to solve the quadratic equation were very successful. Unfortunately, not all students chose this simple technique. Some chose to use the sum of a GP formula and generally got to the correct answer but obviously a more complicated method is more likely to cause careless errors to occur! A common error was the difficulty in finding the correct r value in the $rcis\theta$ form.

(b) Quite well done. A few students left their answer as $\frac{cis\,\pi}{216}$ or similar. Some students ignored the negative part of the index.

QUESTION 28

Too many students did not label the axes or attempt to label the points of intersection. A few students showed their lack of understanding of angles in radian form by drawing $\frac{\pi}{3}$ as a smaller angle than $\frac{\pi}{4}$. Many did not draw their $Arg\ \frac{\pi}{4}$ line through the point $(1, 1)$ or even close.

QUESTION 29

(a) This question was not done well. A large number of students substituted $(z - ki)$ and $(z + ki)$ and got nowhere. Those who did approach this problem correctly often gave their solution as $k = \pm\sqrt{2}, \pm 7, 0$ without ensuring that the solutions satisfied both $Real\ part = 0$ and $Imaginary\ part = 0$.

(b) Some students solved this using their calculators.

Those who did give $k = \pm\sqrt{2}, \pm 7, 0$ as their solution for part (a) often gave $z = \pm\sqrt{2i}, \pm 7i, 0$ as their solution to $P(z) = 0$

QUESTION 30

(a) This question was reasonably well done.

(b) A few students did this question perfectly, however, many did not. Many students went round in circles (incorrectly) e.g.

$$\cos^4\theta = \left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right)^4 = \left(\frac{1}{2}\right)^4 \left(e^{i\theta} + e^{-i\theta}\right)^4 = \left(\frac{1}{2}\right)^4 (\cos\theta)^4 \text{ etc}$$

MATHEMATICS – SPECIALISED (MTS415118)

SOLUTIONS

SECTION A – SEQUENCES AND SERIES

QUESTION 1

$$u_{n+1} = \frac{1}{u_n}\ , u_1 = 2 \qquad \Rightarrow\ u_2 = \frac{1}{2}\ ,\ u_3 = 2\ ,\ u_4 = \frac{1}{2}\ , u_5 = 2\ ,\ u_6 = \frac{1}{2}\ , \ldots$$

So $u_{100} = \dfrac{1}{2}$

QUESTION 2

$$S_n = n(n+2)$$

$$u_n = S_n - S_{n-1}$$

$$u_n = n(n+2) - (n-1)(n+1)$$

$$u_n = n^2 + 2n - n^2 + 1$$

$$u_n = 2n + 1$$

QUESTION 3

(a) $f(x) = \frac{1}{1+2x} \qquad f(0) = 1$

$f'(x) = \frac{-2}{(1+2x)^2} \quad f'(0) = -2$

$f''(x) = \frac{8}{(1+2x)^3} \quad f''(0) = 8$

$f'''(x) = \frac{-48}{(1+2x)^4} \quad f'''(0) = -48$

$$\frac{1}{1+2x} = 1 + (-2)x + 8\frac{x^2}{2!} + (-48)\frac{x^3}{3!} + \cdots$$
$$= 1 - 2x + 4x^2 - 8x^3 + \cdots$$

(b) $\int \frac{1}{1+2x}\,dx = \frac{1}{2}\ln(1+2x) + c$

$$\ln(1+2x) = 2\int \frac{1}{1+2x}\,dx$$

So $= 2\int(1 - 2x + 4x^2 - 8x^3 + \cdots)dx$

$$= 2\left(x - x^2 + \frac{4x^3}{3} - 2x^4 + \cdots\right) + c \quad (\text{assume } \ c = 0)$$

$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \cdots$$

QUESTION 4

Prove: $\lim_{n\to\infty}\left(\frac{n^2}{1+n+n^2}\right) = 1$ i.e. for any $\epsilon > 0, \exists\, N(\epsilon)\ s.t. \left|\frac{n^2}{1+n+n^2} - 1\right| < \epsilon\ \forall\, n > N$

Proof: $\left|\frac{n^2}{1+n+n^2} - 1\right| < \epsilon$

$$\Leftarrow \left|\frac{n^2 - (1+n+n^2)}{1+n+n^2}\right| < \epsilon$$

$$\Leftarrow \left|\frac{-(1+n)}{1+n+n^2}\right| < \epsilon$$

$$\Leftarrow \frac{n+1}{1+n+n^2} < \epsilon$$

$$\Leftarrow \frac{n+1}{n+n^2} < \epsilon \qquad as\ \frac{n+1}{1+n+n^2} < \frac{n+1}{n+n^2}$$

$$\Leftarrow \frac{1}{n} < \epsilon$$

$$\Leftarrow \quad n > \frac{1}{\epsilon} \qquad \text{let } \; N = \frac{1}{\epsilon}$$

$$\text{so } \; \forall\, n > N, \left|\frac{n^2}{1+n+n^2} - 1\right| < \epsilon \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{n^2}{1+n+n^2}\right) = 1$$

OR Prove: $\lim_{n\to\infty}\left(\frac{n^2}{1+n+n^2}\right) = 1$

i.e. for any $\epsilon > 0, \exists\, N(\epsilon)\; s.t. \left|\frac{n^2}{1+n+n^2} - 1\right| < \epsilon \;\; \forall\, n > N$

Proof: $\left|\frac{n^2}{1+n+n^2} - 1\right| < \epsilon$

$$\Leftarrow \left|\frac{n^2 - (1+n+n^2)}{1+n+n^2}\right| < \epsilon$$

$$\Leftarrow \left|\frac{-(1+n)}{1+n+n^2}\right| < \epsilon$$

$$\Leftarrow \frac{n+1}{1+n+n^2} < \epsilon$$

$$\Leftarrow \quad n+1 < \epsilon + \epsilon n + \epsilon n^2$$

$$\Leftarrow \epsilon n^2 + (\epsilon - 1)n + \epsilon - 1 < 0$$

$$\Leftarrow n > \frac{-(\epsilon-1) + \sqrt{(\epsilon-1)^2 - 4\epsilon(\epsilon-1)}}{2\epsilon}$$

$$= \frac{(1-\epsilon) + \sqrt{1 + 2\epsilon - 3\epsilon^2}}{2\epsilon}$$

$$\text{let } N = \frac{(1-\epsilon) + \sqrt{1 + 2\epsilon - 3\epsilon^2}}{2\epsilon}$$

$$\text{so } \; \forall\, n > N, \left|\frac{n^2}{1+n+n^2} - 1\right| < \epsilon$$

$$\text{and } \; \lim_{n\to\infty}\left(\frac{n^2}{1+n+n^2}\right) = 1$$

QUESTION 5

(a) $\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{2r+1-(2r-1)}{(2r-1)(2r+1)} = \frac{2}{4r^2-1}$

(b)

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{1}{2}\sum_{r=1}^{n} \frac{2}{4r^2-1}$$

$$= \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

$1 \; (\; 2n \;)$

(c)

$$\sum_{r=100}^{\infty} \frac{1}{4r^2-1} = \sum_{r=1}^{\infty} \frac{1}{4r^2-1} - \sum_{r=1}^{99} \frac{1}{4r^2-1}$$

$$= \lim_{n\to\infty}\left(\frac{n}{2n+1}\right) - \frac{99}{2\times 99+1}$$

$$= \frac{1}{2} - \frac{99}{199}$$

$$= \frac{1}{398}$$

QUESTION 6

(a) $S_4 = 4^2 - 3^2 + 2^2 - 1^2 = 10$

(b) $S_{n+1} = (n+1)^2 - n^2 + (n-1)^2 - (n-2)^2 + (n-3)^2 + \cdots + (-1)^n$

$$(n+1)^2 - S_n$$
$$= (n+1)^2 - (n^2 - (n-1)^2 + (n-2)^2 - (n-3)^2 + \cdots + (-1)^{n-1})$$
$$= (n+1)^2 - n^2 + (n-1)^2 - (n-2)^2 + (n-3)^2 + \cdots - (-1)^{n-1}$$
$$= (n+1)^2 - n^2 + (n-1)^2 - (n-2)^2 + (n-3)^2 + \cdots + (-1)^n$$
$$= S_{n+1}$$

(c) Prove: $n^2 - (n-1)^2 + (n-2)^2 - (n-3)^2 + \cdots + (-1)^{n-1} = \frac{n(n+1)}{2}, n\epsilon\mathbb{Z}^+$

Proof: Let P_n be the statement above

Consider P_1 $\quad LHS = 1^2 = 1$

$$RHS = \frac{1(1+1)}{2} = 1 = LHS$$

So P_1 is true

Assume P_k is true

$$i.e.\ \ k^2 - (k-1)^2 + (k-2)^2 - (k-3)^2 + \cdots + (-1)^{k-1} = \frac{k(k+1)}{2}, k\epsilon\mathbb{Z}^+$$

Consider P_{k+1}

$$i.e.\ \ (k+1)^2 - k^2 + (k-1)^2 - (k-2)^2 + \cdots + (-1)^k = \frac{(k+1)(k+2)}{2}, k\epsilon\mathbb{Z}^+$$

$LHS = \quad (k+1)^2$

$-\frac{\mathrm{k(k+1)}}{2}$ from part (b) and assuming P_k is true

$= \frac{2(k+1)^2 - k(k+1)}{2}$

$= \frac{k^2 + 3k + 2}{2}$

$= \frac{(k+1)(k+2)}{2} = RHS$

So P_k true $\Longrightarrow P_{k+1}$ true

Since P_1 is true and P_k true $\Longrightarrow P_{k+1}$ true then by the Principle of Mathematical Induction P_n is true $\forall n \epsilon \mathbb{Z}^+$.

SECTION B – MATRICES AND LINEAR TRANSFORMATIONS

QUESTION 7

(a) $A^2 = \begin{pmatrix} 2 & 0 \\ 1 & 8 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 10 & 64 \end{pmatrix}$

(b) $\det A = 2 \times 8 - 0 \times 1 = 16$

$$A^{-1} = \frac{1}{16}\begin{pmatrix} 8 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{16} & \frac{1}{8} \end{pmatrix}$$

QUESTION 8

(a) $|A| = ad - bc$

$|A^T| = ad - cb = |A|$

(b) $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$RR^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

QUESTION 9

(a) $y = \frac{1}{\sqrt{3}}x$ so $\tan\theta = \frac{1}{\sqrt{3}}$ $\therefore \theta = \frac{\pi}{6}$ $\Rightarrow$

$$\begin{pmatrix} \cos 2\frac{\pi}{6} & \sin 2\frac{\pi}{6} \\ \sin 2\frac{\pi}{6} & -\cos 2\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$x' = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$ and $y' = \frac{\sqrt{3}}{2}x - \frac{1}{2}y$

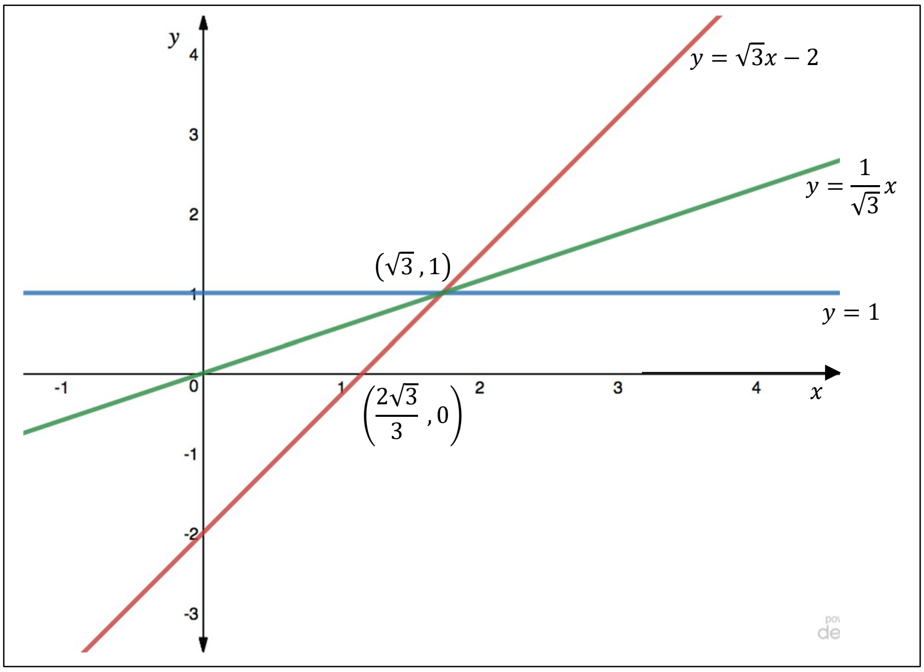
image line is $x'\sqrt{3} - y' = 2$

So $\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)\sqrt{3} - \left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right) = 2$

$$\frac{\sqrt{3}}{2}x + \frac{3}{2}y - \frac{\sqrt{3}}{2}x + \frac{1}{2}y = 2$$

$$2y = 2 \Rightarrow y = 1$$

(b)



QUESTION 10

(a) $$\begin{pmatrix} 1 & -2 & 2 & 1 \\ -1 & 3 & -1 & k \\ 2 & -5 & k & -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & 1 \\ 0 & 1 & 1 & k+1 \\ 0 & -1 & k-4 & -4 \end{pmatrix} \begin{matrix} \\ R_1 + R_2 \\ R_3 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 1 & -2 & 2 & 1 \\ 0 & 1 & 1 & k+1 \\ 0 & 0 & k-3 & k-3 \end{pmatrix} R_3 + R_2$$

$$= \begin{pmatrix} 1 & -2 & 2 & 1 \\ 0 & 1 & 1 & k+1 \\ 0 & 0 & 1 & 1 \end{pmatrix} R_3 \div (k-3), k \neq 3$$

$$= \begin{pmatrix} 1 & -2 & 2 & 1 \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 1 \end{pmatrix} R_2 - R_3$$

$$= \begin{pmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 1 \end{pmatrix} R_1 - 2R_3$$

$$= \begin{pmatrix} 1 & 0 & 0 & 2k-1 \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 1 \end{pmatrix} R_1 + 2R_2$$

(b) for $k \neq 3$, $x = 2k - 1$, $y = k$, $z = 1$

(c) for $k = 3$, matrix becomes $\begin{pmatrix} 1 & -2 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

let $z = t$, $y + z = 4 \Rightarrow y = 4 - t$

$x - 2y + 2z = 1 \Rightarrow x = 9 - 4t$

i.e. the 3 planes meet in a straight line given by $x = 9 - 4t$, $y = 4 - t$, $z = t$

QUESTION 11

(a) Let the plane be $ax + by + cz = 1$

$(1, 4, 2) \Rightarrow a + 4b + 2c = 1$

$(3, 2, 1) \Rightarrow 3a + 2b + c = 1$

$(5, 3, 5) \Rightarrow 5a + 3b + 5c = 1$

$$\begin{pmatrix} 1 & 4 & 2 & 1 \\ 3 & 2 & 1 & 1 \\ 5 & 3 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & {}^{1}/_{5} \\ 0 & 1 & 0 & {}^{2}/_{7} \\ 0 & 0 & 1 & -{}^{6}/_{35} \end{pmatrix}$$

$a = \frac{1}{5}, \ b = \frac{2}{7}, \ c = -\frac{6}{35}$

So $\frac{1}{5}x + \frac{2}{7}y - \frac{6}{35}z = 1$

$\Rightarrow \ 7x + 10y - 6z = 35$

(b) $x = 1 + 2t, \ y = 1 + t, \ z = -3 + 4t$

$$\begin{aligned} \Rightarrow 7x + 10y - 6z &= 7(1 + 2t) + 10(1 + t) - 6(-3 + 4t) \\ &= 7 + 14t + 10 + 10t + 18 - 24t \\ &= 35 \quad \text{as required} \end{aligned}$$

So the plane $7x + 10y - 6z = 35$ contains the line given by

$x = 1 + 2t, \ y = 1 + t, \ z = -3 + 4t$

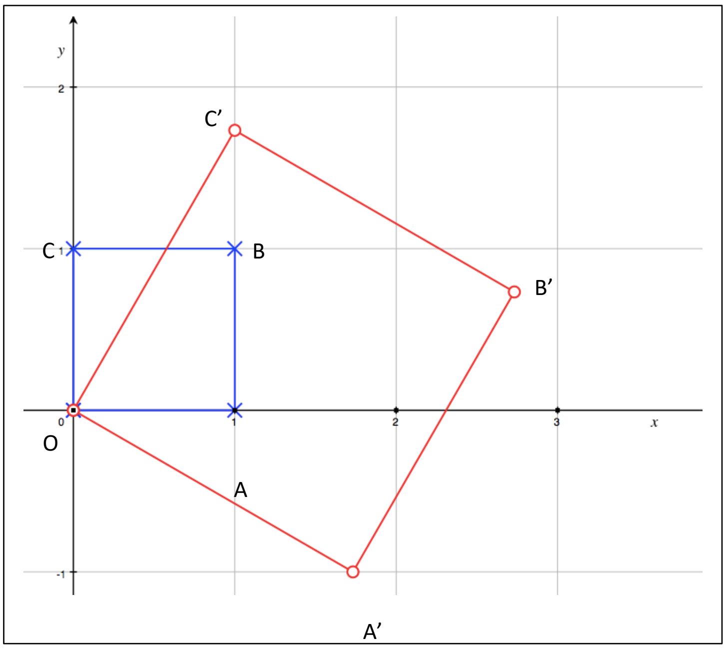
QUESTION 12

(a) $\begin{pmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = O'$

$\begin{pmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} = A'$

$\begin{pmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\sqrt{3}+1\\\sqrt{3}-1\end{pmatrix} = B'$

$\begin{pmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = C'$



(b) ***M*** is a dilation by a factor of 2 in both the x and the y directions, followed by a rotation clockwise through $\frac{\pi}{6}$ radians (by inspection!)

Dilation $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$

Rotation $\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{-1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$

OR using an algebraic approach .....

dilation by a factor of a in the x direction and by a factor of b in the y direction $\Rightarrow \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$

Rotation matrix $\Rightarrow \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

$$\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}a\cos\theta & -a\sin\theta\\ b\sin\theta & b\cos\theta\end{pmatrix} = \begin{pmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{pmatrix}$$

$$\therefore\ a\cos\theta = \sqrt{3} \ \text{ and } \ -a\sin\theta = 1 \quad \Rightarrow \tan\theta = -\frac{1}{\sqrt{3}} \ \text{ so } \ \theta = -\frac{\pi}{6}$$

$$a\cos\left(-\frac{\pi}{6}\right) = \sqrt{3} \quad \Rightarrow a = \frac{\sqrt{3}}{\sqrt{3}/2} = 2$$

$$b\sin\left(-\frac{\pi}{6}\right) = -1 \quad \Rightarrow b = \frac{-1}{-1/2} = 2$$

So ***M*** is a dilation by a factor of 2 in both the x and the y directions, followed by a rotation clockwise through $\frac{\pi}{6}$ radians.

(c) $\boldsymbol{K} = \begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}$

K has a similar effect to ***M*** = dilation by a factor of $\sqrt{k^2+1}$ in both the x and the y

directions and a rotation (clockwise through $\arctan\left(\frac{1}{k}\right)$)

Original circle $x^2 + y^2 = 1$ circle with centre $(0,0)$ and $r = 1$

$\longrightarrow$ image $x^2 + y^2 = k^2 + 1$ circle with centre $(0,0)$ and $r = \sqrt{k^2+1}$

SECTION C – DIFFERENTIAL CALCULUS AND AREA/VOLUME

QUESTION 13

$$y = \arctan x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{-2x}{(1+x^2)^2}$$

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = (1+x^2)\cdot\frac{-2x}{(1+x^2)^2} + 2x\cdot\frac{1}{1+x^2}$$

$$= \frac{-2x}{1+x^2} + \frac{2x}{1+x^2}$$

$$= 0 \quad \text{as required}$$

QUESTION 14

$$x^2y^2 = 1$$

$$2xy^2 + x^2\cdot 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2xy^2}{2x^2y} = -\frac{y}{x}$$

$$\frac{d^2y}{dx^2} = \frac{-1\frac{dy}{dx}\cdot x - (-y)\cdot 1}{x^2}$$

$$= \frac{y - \frac{dy}{dx}\cdot x}{x^2}$$

$$= \frac{y - \left(-\frac{y}{x}\right)\cdot x}{x^2}$$

$$= \frac{2y}{x^2} \qquad \text{as required}$$

QUESTION 15

(a) $\cos y + y \sin x = x^2$

If $x = -1$ and $y = 0$ then $\cos y + y \sin x = \cos 0 + 0 \sin(-1) = 1$

and $x^2 = (-1)^2 = 1$

If $x = 0$ and $y = \frac{\pi}{2}$ then $\cos y + y \sin x = \cos\frac{\pi}{2} + \frac{\pi}{2}\sin 0 = 0$

and $x^2 = (0)^2 = 0$

So $(-1, 0)$ and $\left(0, \frac{\pi}{2}\right)$ both lie on the curve.

(b) $\cos y + y \sin x = x^2$

$$-\sin y \frac{dy}{dx} + \frac{dy}{dx}\sin x + y\cos x = 2x$$

$$\left(0, \frac{\pi}{2}\right) \Rightarrow -\sin\frac{\pi}{2}\cdot\frac{dy}{dx} + \frac{dy}{dx}\sin 0 + \frac{\pi}{2}\cos 0 = 0$$

$$\frac{dy}{dx} = \frac{\pi}{2}$$

So tangent has gradient of $\frac{\pi}{2}$

$$(-1, 0) \text{ and } \left(0, \frac{\pi}{2}\right) \quad \Rightarrow m = \frac{\frac{\pi}{2} - 0}{0 - (-1)} = \frac{\pi}{2}$$

Same gradient with a common point so the tangent to the curve at $\left(0, \frac{\pi}{2}\right)$ also passes through $(-1, 0)$.

QUESTION 16

(a) $\frac{d}{dx}\left(2^{\sqrt{1-x}}\right) = 2^{\sqrt{1-x}} \cdot \ln 2 \cdot -\frac{1}{2}(1-x)^{-\frac{1}{2}}$

$$= \frac{-2^{\sqrt{1-x}} \ln 2}{2\sqrt{1-x}}$$

(b) $y = \frac{2^{\sqrt{1-x}}}{\sqrt{1-x}}, -3 \leq x \leq 0$

$$area = \int_{-3}^{0} \frac{2^{\sqrt{1-x}}}{\sqrt{1-x}} dx$$

$$= \frac{-2}{\ln 2} \int_{-3}^{0} \frac{-2^{\sqrt{1-x}} \ln 2}{2\sqrt{1-x}} dx$$

$$= \frac{-2}{\ln 2} \left[2^{\sqrt{1-x}}\right]_{-3}^{0}$$

$$= \frac{-2}{\ln 2} (2 - 2^2)$$

$$= \frac{4}{\ln 2}\ units^2$$

QUESTION 17

(a) $f(x) = x(\ln x)^2, x > 0$

$$f'(x) = (\ln x)^2 + x \cdot 2\ln x \cdot \frac{1}{x} = (\ln x)^2 + 2\ln x = \ln x(\ln x + 2)$$

$$f''(x) = 2\ln x \cdot \frac{1}{x} + \frac{2}{x} = \frac{2}{x}(\ln x + 1)$$

Stat pts when $f'(x) = 0$

$\ln x(\ln x + 2) = 0$

$\ln x = 0$ and $\ln x + 2 = 0$

$x = 1$ and $x = e^{-2}$

at $x = 1$, $f(1) = 1\,(\ln 1)^2 = 0$, $f'(1) = 0$, $f''(1) = \frac{2}{1}\,(\ln 1 + 1) = 2 > 0$ so concave up

$\therefore$ $(1, 0)$ is a local min pt

at $x = e^{-2}$, $f(e^{-2}) = e^{-2}\,(\ln e^{-2})^2 = 4e^{-2}$, $f'(e^{-2}) = 0$

$$f''(e^{-2}) = \frac{2}{e^{-2}}\,(\ln e^{-2} + 1) = -2e^{-2} < 0 \quad \text{so concave down}$$

$\therefore$ $(e^{-2}, 4e^{-2})$ is a local max pt

Pts of inflection when $f''(x) = 0$

$$\frac{2}{x}\,(\ln x + 1) = 0$$

$\frac{2}{x} = 0$ and $\ln x + 1 = 0$

no soln and $x = e^{-1}$

at $x = e^{-1}$, $f(e^{-1}) = e^{-1}\,(\ln e^{-1})^2 = e^{-1}$

$$f'(e^{-1}) = \ln e^{-1}\,(\ln e^{-1} + 2) = -1 < 0$$

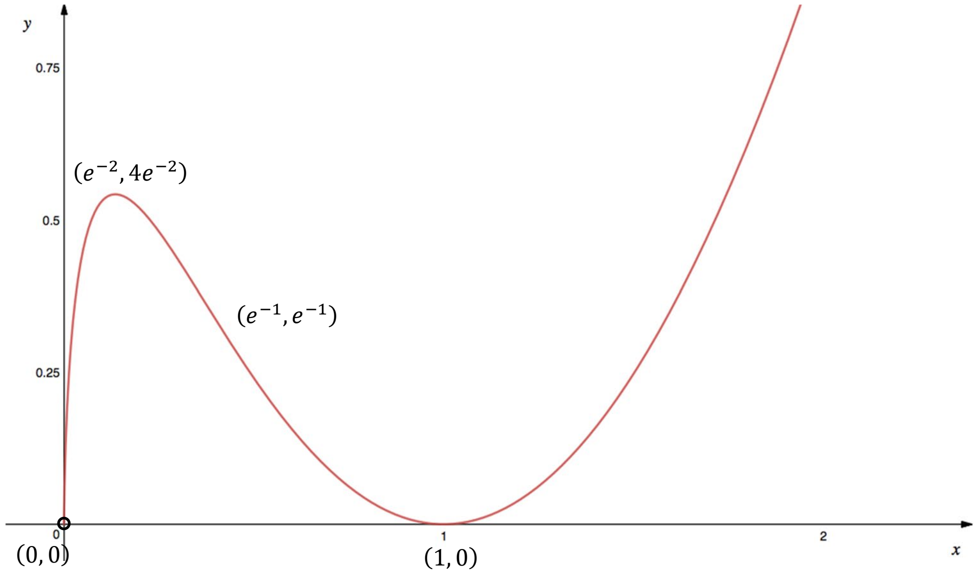
$$f''(e^{-1}) = \frac{2}{e^{-1}}\,(\ln e^{-1} + 1) = 0$$

x	0.2	e^{-1}	0.5
$f''(x)$	< 0	$= 0$	> 0

as $x \uparrow e^{-1}, f''(x)$ changes from concave down to concave up

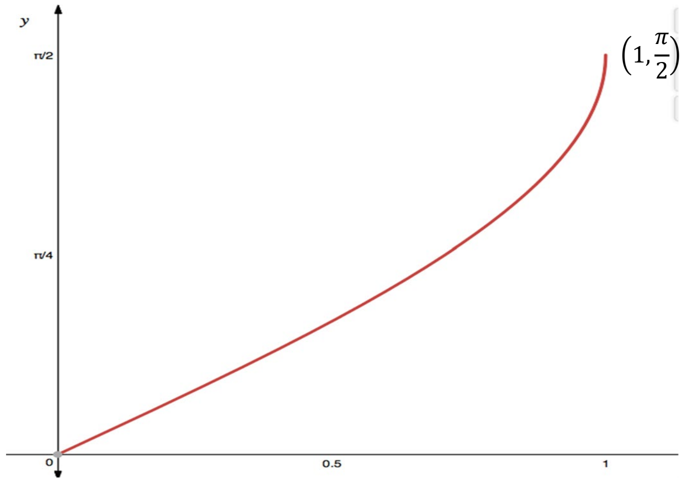
$\therefore$ (e^{-1}, e^{-1}) is a pt of inflection (decreasing curve)

(b) $f(x) = x\,(\ln x)^2\,, x > 0$



QUESTION 18

(a) $y = \arcsin x\,,\ 0 \le x \le 1$



(b)

$$area = \int_0^1 \arcsin x\ dx \qquad \text{let } f = \arcsin x \ \Rightarrow f'$$

$$= \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad g' = 1 \ \Rightarrow g = x$$

$$= [\ x \arcsin x\ ]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} + \frac{1}{2}\int_{1}^{0} u^{-\frac{1}{2}} dx \qquad let\ u = 1 - x^2 \Rightarrow du$$

$$= -2x\, dx \qquad x = 0 \Rightarrow u = 1$$

$$-\frac{1}{2}du = x\, dx \qquad x = 1$$

$$\Rightarrow u = 0$$

$$= \frac{\pi}{2} + \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_1^0$$

$$= \frac{\pi}{2} + \frac{1}{2}(0 - 2)$$

$$= \frac{\pi}{2} - 1\ units^2$$

(c)

$$volume = \pi\int_0^{\frac{\pi}{2}} 1^2 dy - \pi\int_0^{\frac{\pi}{2}} (\sin y)^2 dy \qquad y = \arcsin x \Rightarrow x$$

$$= \sin y$$

$$= \pi\,[y]_0^{\frac{\pi}{2}} - \pi\int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 2y\right)^2 dy \qquad \cos 2y = 1 - 2\sin^2 y$$

$$= \pi\left(\frac{\pi}{2} - 0\right) - \pi\left[\frac{y}{2} - \frac{1}{4}\sin 2y\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{2} - \pi\left(\left(\frac{\pi}{4} - \frac{1}{4}\sin\pi\right) - \left(0 - \frac{1}{4}\sin 0\right)\right)$$

$$= \frac{\pi^2}{2} - \frac{\pi^2}{4}$$

$$= \frac{\pi^2}{4}\ units^3$$

SECTION D – INTEGRAL CALCULUS

QUESTION 19

$$\frac{x+3}{x\,(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$x + 3 = A(x - 1) + Bx$

let $x = 0 \quad 3 = -A \;\Rightarrow A = -3$

let $x = 1 \quad 4 = B$

$$\int \frac{x+3}{x\,(x-1)}\,dx = \int \frac{4}{x-1} - \frac{3}{x}\,dx$$

$$= 4\ln|x-1| - 3\ln|x| + c$$

$$= \ln\left|\frac{(x-1)^4}{x^3}\right| + c$$

QUESTION 20

$$\int_1^e x^3 \ln x\; dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \frac{1}{4}\int_1^e x^3\,dx$$

let $f(x) = \ln x \quad f'(x) = \frac{1}{x}$

$g'(x) = x^3 \quad g(x) = \frac{x^4}{4}$

$$= \frac{e^4}{4}\ln e - 0 - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^e$$

$$= \frac{e^4}{4} - \frac{1}{4}\left(\frac{e^4}{4} - \frac{1}{4}\right)$$

$$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{4}$$

$$= \frac{3e^4+1}{4}$$

QUESTION 21

(a) $y = x\sqrt{1-x^2} - \arccos x$

$$\frac{dy}{dx} = \sqrt{1-x^2} + x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x - \frac{1}{-\sqrt{1-x^2}}$$

$$= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$= \sqrt{1-x^2} + \frac{1-x^2}{\sqrt{1-x^2}}$$

$$= 2\sqrt{1-x^2}$$

(b)

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{2}\int_0^1 2\sqrt{1-x^2}\, dx$$

$$= \frac{1}{2}\left[x\sqrt{1-x^2} - \arccos x\right]_0^1$$

$$= \frac{1}{2}\left(0 - \left(0 - \frac{\pi}{2}\right)\right)$$

$$= \frac{\pi}{4}$$

QUESTION 22

(a) $\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{t^2}$

$$\int A^{-\frac{3}{2}} dA = \int t^{-2} dt$$

$$-2A^{-\frac{1}{2}} = -\frac{1}{t} + c$$

$$\frac{2}{\sqrt{A}} = \frac{1}{t} + c$$

$$t = 1, A = 1 \quad \Rightarrow \quad 2 = 1 + c \quad \therefore c = 1$$

So $\frac{2}{\sqrt{A}} = \frac{1}{t} + 1$

$$\frac{2}{\sqrt{A}} = \frac{1+t}{t}$$

$$\sqrt{A} = \frac{2t}{t+1}$$

$$A = \left(\frac{2t}{t+1}\right)^2$$

(b) $\frac{dA}{dt} > 0$, so A is increasing for all t

$as\ t \to \infty, \frac{2t}{t+1} \to 2 \qquad$ so $A \to 2^2 = 4km^2$

QUESTION 23

(a) Method 1

let $x = \tan u \Rightarrow dx = \sec^2 u \; du$

$x = 1 \Rightarrow u = \frac{\pi}{4}$

$x = \sqrt{3} \Rightarrow u = \frac{\pi}{3}$

$$\int_1^{\sqrt{3}} \frac{1}{x\,(1+x^2)}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 u \; du}{\tan u \;(1+\tan^2 u\,)}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{du}{\tan u}$$

with $a = \frac{\pi}{4}$ and $b = \frac{\pi}{3}$

Method 2

let $x = \tan u \Rightarrow u = \arctan x$ so $du = \frac{dx}{1+x^2}$

$x = 1 \Rightarrow u = \frac{\pi}{4}$

$x = \sqrt{3} \Rightarrow u = \frac{\pi}{3}$

$$\int_1^{\sqrt{3}} \frac{1}{x\,(1+x^2)}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{du}{\tan u}$$

with $a = \frac{\pi}{4}$ and $b = \frac{\pi}{3}$

(b)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{du}{\tan u} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos u}{\sin u}\,du$$

let $t = \sin u \Rightarrow dt = \cos u \; du$

$u = \frac{\pi}{3} \Rightarrow t = \frac{\sqrt{3}}{2}$ *and* $u = \frac{\pi}{4} \Rightarrow t = \frac{\sqrt{2}}{2}$

$$= \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{t}\,dt$$

$$= [\ln t]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{2}}{2}$$

$$= \ln\frac{\sqrt{3}}{\sqrt{2}}$$

$$= \frac{1}{2}\ln\frac{3}{2}$$

QUESTION 24

$$xy\frac{dy}{dx} = x^2 + 2y^2$$

$$\frac{dy}{dx} = \frac{x^2 + 2y^2}{xy} \qquad let\ V = \frac{y}{x} \Rightarrow y = Vx\ \ and\ \ \frac{dy}{dx} = V + x\frac{dV}{dx}$$

$$V + x\frac{dV}{dx} = \frac{x^2 + 2V^2x^2}{Vx^2}$$

$$= \frac{1 + 2V^2}{V}$$

$$x\frac{dV}{dx} = \frac{1 + V^2}{V}$$

$$\int \frac{V}{1 + V^2}\,dV = \frac{1}{2}\int \frac{1}{u}\,du$$

where $u = 1 + V^2$ and $du = 2V dV$

$$\int \frac{V}{1 + V^2}\,dV = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\ln(1 + V^2) = \ln x + c$$

$$\frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = \ln x + c$$

$$(1,0) \Rightarrow \frac{1}{2}\ln\left(1 + \frac{0^2}{1^2}\right) = \ln 1 + c$$

$$\therefore c = 0$$

$$\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln x$$

$$\ln\left(1+\frac{y^2}{x^2}\right) = 2\ln x = \ln x^2$$

$$1+\frac{y^2}{x^2} = x^2$$

$$y^2 = x^2\,(x^2-1)$$

$$y = \pm\sqrt{x^2(x^2-1)} = \pm\, x\,\sqrt{(x^2-1)}$$

SECTION E – COMPLEX NUMBERS

QUESTION 25

$z = 1 - i$

(a) $z^2 = (1-i)^2 = 1 - 2i + i^2 = -2i$

which is purely imaginary

(b) $(z^2)^2 = (-2i)^2 = 4i^2 = -4$ which is real

So $n = 4$ is a positive integer such that z^n is real

QUESTION 26

$z = a + bi$ for $a, b \epsilon \mathbb{R}$

$$\frac{1}{z} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$$

$$\frac{\bar{z}}{|z|^2} = \frac{a-bi}{\left(\sqrt{a^2+b^2}\right)^2} = \frac{a-bi}{a^2+b^2} = \frac{1}{z}$$

QUESTION 27

(a) $z^2 - 6z + 36 = 0$

$$z = \frac{6 \pm \sqrt{6^2 - 4 \times 1 \times 36}}{2 \times 1}$$

$$z = \frac{6 \pm \sqrt{-108}}{2} = \frac{6 \pm 6\sqrt{3}\,i}{2} = 3 \pm 3\sqrt{3}\,i$$

$$z = 3 + 3\sqrt{3}\,i = 6cis\left(\frac{\pi}{3}\right)$$

$$z = 3 - 3\sqrt{3}\,i = 6cis\left(-\frac{\pi}{3}\right)$$

(b) $z^{-3} = \left(6cis\left(\pm\frac{\pi}{3}\right)\right)^{-3}$

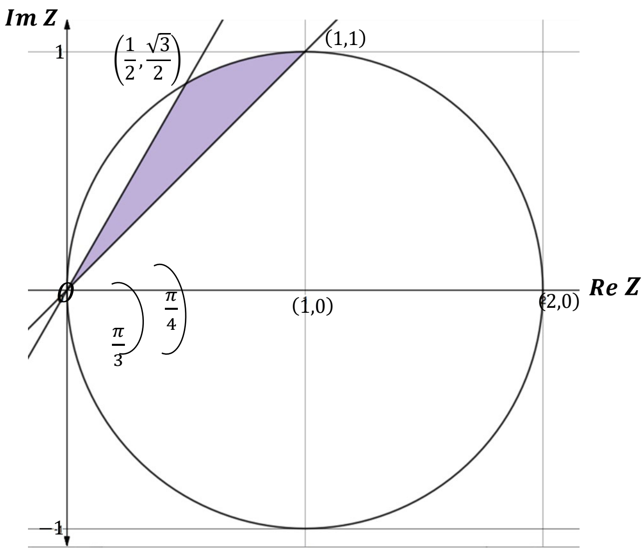
$= 6^{-3}cis\left(\mp\frac{3\pi}{3}\right)$

$= \frac{1}{216}cis(\mp\pi)$

$= -\frac{1}{216}$

QUESTION 28

$\left\{Z: \frac{\pi}{4} \leq Arg\ Z \leq \frac{\pi}{3}\right\} \cap \{Z: |Z-1| \leq 1\}$



QUESTION 29

(a) $P(z) = z^4 - 2z^3 + 51z^2 - 98z + 98$

$P(ki) = (ki)^4 - 2(ki)^3 + 51(ki)^2 - 98ki + 98 = 0$

$\Rightarrow k^4 + 2k^3i - 51k^2 - 98ki + 98 = 0$

$(k^4 - 51k^2 + 98) + i(2k^3 - 98k) = 0$

So, real: $k^4 - 51k^2 + 98 = 0$ imaginary: $2k^3 - 98k = 0$

$(k^2 - 49)(k^2 - 2) = 0$ $2k(k^2 - 49) = 0$

$$k = \pm 7, \pm\sqrt{2} \qquad\qquad k = 0, \pm 7$$

$$\therefore k = \pm 7 \quad \text{(satisfies both conditions)}$$

(b) from part (a) we know that $z - 7i$ and $z + 7i$ are both factors

$$(z - 7i)(z + 7i) = z^2 + 49$$

$$P(z) = z^4 - 2z^3 + 51z^2 - 98z + 98$$

$$= (z^2 + 49)(z^2 + az + b)$$

$$= z^4 + az^3 + (b + 49)z^2 + 49az + 49b$$

Comparing coefficients gives

$$a = -2 \quad \text{and} \quad b + 49 = 51 \Rightarrow b = 2$$

$$\text{So} \quad P(z) = (z^2 + 49)(z^2 - 2z + 2)$$

$$= (z - 7i)(z + 7i)(z - 1 - i)(z - 1 + i)$$

$$P(z) = 0 \Rightarrow z = \pm 7i, 1 \pm i$$

QUESTION 30

(a) $\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta))$

$$= \frac{1}{2}(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)$$

$$= \frac{1}{2}(2\cos\theta)$$

$$= \cos\theta$$

(b) $\cos^4\theta = \left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right)^4$

$$= \frac{1}{16}\left(\left(e^{i\theta}\right)^4 + 4\left(e^{i\theta}\right)^3 e^{-i\theta} + 6\left(e^{i\theta}\right)^2\left(e^{-i\theta}\right)^2 + 4e^{i\theta}\left(e^{-i\theta}\right)^3 + \left(e^{-i\theta}\right)^4\right)$$

$$= \frac{1}{16}\left(e^{i4\theta} + 4e^{i3\theta}\, e^{-i\theta} + 6e^{i2\theta}e^{-i2\theta} + 4e^{i\theta}e^{-i3\theta} + e^{-i4\theta}\right)$$

$$= \frac{1}{16}\left(e^{i4\theta} + 4e^{i2\theta} + 6 + 4e^{-i2\theta} + e^{-i4\theta}\right)$$

$$= \frac{1}{16}\left(\left(e^{i4\theta} + e^{-i4\theta}\right) + 4\left(e^{i2\theta} + e^{-i2\theta}\right) + 6\right)$$

$$= \frac{1}{16}(2\cos 4\theta + 4(2\cos 2\theta) + 6)$$

$$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

as required