MATHEMATICS - METHODS (MTM415117)

PART 1: CALCULATORS NOT PERMITTED

Candidates continue to lose marks due to poor skills in managing fractions, decimals and directed numbers. Given this exam has a calculator-free section, these skills are an expectation of this course. Lack of skills in basic algebra processes is still costing candidates marks.

Note: These comments have been consistently made over recent years.

PART 1 SECTION A FUNCTION STUDY (C4)

QUESTION 1

Generally well handled with most candidates expanding $(ax + b)^3$ and equating the coefficients of x^3 and the constants to find *a* and *b*. Common errors include: using ax^3 instead of $(ax)^3$, ax^2 instead of $(ax)^2$ etc…, when expanding $(ax + b)^3$.

QUESTION 2

Most candidates knew to use $P(-1) = 5$ and $P(1) = -3$ to set up 2 equations in *a* and *b*, and solved these simultaneously. However, algebraic and calculation errors, failure to simplify equations and poor setting out, lead to errors.

Some candidates did not use the remainder theorem, instead opting for algebraic long division. This was a time consuming approach and few were successful.

QUESTION 3

Most candidates knew how to find zeros and solved $f(x) = 0$. However, some candidates did not understand the term "zero" and found the *y*-intercept instead.

Again algebraic and calculation errors were common.

QUESTION 4

Candidates found this question difficult with many unable to rewrite 16^x as $(4^x)^2$ to complete the required substitution and get the quadratic, $2A^2 + 3A - 2 + 0$. Many candidates found factorising the quadratic difficult. Once factorised, most candidates were able to find the required solution and realised $4^x = -2$ or $log_4(-2)$ was undefined.

QUESTION 5

a) Generally well done. A common error was incorrect notation with writing domain and range.

b) Most knew how to approach this question, although some simply replaced *h* with 4 in the equation without stating or explaining that *h*=4.

Many candidates did not know that, $log_2 1 = 0$, although this is on their formula sheet. Similarly many could not simplify, $log_2 4$.

PART 1 SECTION B TRIGONOMETRY (C5)

QUESTION 6

a) This question was done very well. Arithmetic problems including simplifying fractions were the main reasons for errors made.

b) This part was also done well. Common errors were not drawing in an angle that is approximately 75 degrees or choosing the entirely wrong quadrant to place the angle. Some candidates treated a right angle between the *x* and *y* axes as π radians instead of π/2 radians leading to an incorrect placement of the angle expected.

QUESTION 7

This question was done well. Two common approaches were to use the trig identity $\sin^2 x + \cos^2 x = 1$ or to use a right angled triangle with the appropriate sides to represent the sin ratio often with a supplementary CAST diagram. The main errors were incorrect use of Pythagoras' Theorem and incorrect signs for the ratios.

QUESTION 8

This question was done well. Many candidates correctly identified π/4 as the basic angle. Common errors were to subtract rather than add a π/4 term and often solutions outside the domain were included.

QUESTION 9

a) This question was done well, but care needed to be taken in graphing through the appropriate grid points. There was an expectation that required missing minimum and maximum points would be labelled each with an x and y coordinate as was requested in the question. A number of candidates left this question blank which may have been due to them skipping over it when they were quickly reading the page, assuming that the intentional gap in the graph was intended to remain blank.

b) This question was very well done in (i), (ii) and (iii). However, the most common error was not putting a minus sign in front of the horizontal translation, possibly due to misreading the general form given in the question.

c) This question was done well, but a few candidates did not substitute in the value for b obtained in part (b)(iv). Some candidates did not show adequate reasoning for why the y coordinate was 5.

PART 1 SECTION C DIFFERENTIAL CALCULUS (C6)

A commonly occurring flaw in candidate solutions was the lack of brackets when multiplying by a term. For example $(4 - x) \times 8x + 4$ would be erroneously written rather than the correct $(4 - x) \times (8x + 4)$.

QUESTION 10

This question was either well done (by most) or poorly done.

a) was correctly evaluated by substituting $x = -2$.

b) was correctly evaluated by cancelling an $(x - 7)$ factor between the numerator and denominator then substituting $x = 7$. Some incorrectly `factorised' (the already factorised) $(x - 7)^2$ as $(x - 7)(x + 7)$.

QUESTION 11

Many candidates ignored the suggestion to leave answers unsimplified at the cost of time and often accuracy (that is the unsimplified answer was correct and this was then rendered incorrect with simplification/algebra errors).

a) Well done by most candidates who successfully applied the product rule.

b) Candidates had a little more difficulty with this question. Most employed the chain rule but many had the incorrect value of a in the expression $\frac{dy}{dx} = a \times \frac{1}{x} \times e^{4\ln(2x)}$ ($a = 4$ is correct) due to difficulty dealing with $\frac{d(\ln(2x))}{dx} = \frac{1}{2x} \times \frac{d(2x)}{dx} = \frac{1}{2x} \times 2 = \frac{1}{x}$ (in fact replace 2 with any constant k and the same result follows). An infrequently (and often successfully) used approach was to simplify the function prior to differentiating: $y = e^{4\ln(2x)} = \left(e^{\ln(2x)}\right)^4 = (2x)^4$ so $\frac{dy}{dx} = 4 \times 2 \times (2x)^3$

c) Reasonably well done by candidates who employed the quotient rule although some ill-advised simplification was attempted.

QUESTION 12

a) Many candidates (unnecessarily) expanded $4 - (x-5)^2$ (risking algebraic error) to $-x^2 + 10x - 21$ or eventual arithmetic error if $(-6)^2$ is evaluated rather than $-(6)^2$. Another frequent error was the attempt to simplify

$$\sqrt{4-(x-5)^2} \text{ as } \sqrt{4} - \sqrt{(x-5)^2}.$$

Yet another error was interpreting $\sqrt{A}$ as $\pm\sqrt{A}$. The $\sqrt{\ }$ is by definition the positive square root, which needs to be distinguished from the solutions to $x^2 = A$ which gives $x = \pm\sqrt{A}$.

An infrequent but regrettable error was;

$$\frac{d}{dx}\left(\sqrt{4-(x-5)^2} + 2\right) = \frac{d}{dx}\left(\sqrt{4-(x-5)^2}\right) + 2 .$$

The derivative of any constant is zero.

Successfully applying the chain rule to differentiate $\frac{d}{dx}\left(\sqrt{4-(x-5)^2}+2\right)$ whilst understood was error prone.

b) Most attempts recognised the 3 elements using

$y - y_1 = m(x - x_1)$ [or $y = mx + c$]

with m found via $m_T \times m_N = -1$, and

$(x_1, y_1) = (6, y(6))$ [or using this point to find c]

Many candidates when substituting for $-y_1$ wrote $-\sqrt{2}+3$ instead of $-(\sqrt{2}+3)$. The correct m_N seemed to be used regardless of the answer to part a.

c) The cleanest approach to c was substituting $x = 5$ into the LHS of equation and arriving at $y = 2$.

PART 1 SECTION D INTEGRAL CALCULUS (C7)

QUESTION 13

a) Generally done well.

b) Generally done well. A number of candidates did not correctly integrate $\frac{3}{x}$, and also the numerical manipulation proved troublesome.

QUESTION 14

a) Generally done well, but some candidates did not show an intermediate, simplifying step, after substituting.

b) Not done well. Most were able to give the correct opening statement along the lines of $\int_0^\pi g(x) - f(x)dx$ but then most struggled to properly expand and integrate $\frac{4}{\pi^2}x(\pi - x)$, and then also had difficulties correctly substituting and simplifying to get the correct answer.

QUESTION 15

a) Many did not see ln(2) as a constant. When quotient rule was applied it was often done so incorrectly (for instance ln(2) commonly differentiated to ½). There were some very strange approaches and uses of 'logic'.

b) Many were able to complete the integration and substitution, but then struggled to resolve ln(4) into a useful form to enable it to combine with ln(2).

In general for "show" questions a clarity of steps was not always present.

PART 1 SECTION E PROBABILITY (C8)

QUESTION 16:

a) This question was answered well by all candidates.

b) While most candidates had no trouble with this question, its answer was the nemesis of many for Part (c).

A number of others made errors calculating with fractions which then prevented them from gaining full marks.

c) Most candidates had the correct formula for Var(X), but a common error was to misread the expected value needed for the substitution (using A = 5 rather than the E(X) = 2 from part b). Candidates may find it useful to highlight key features of a question to ensure that the intended focus of the question is utilised.

d) Most candidates calculated the correct answer (-100) which was a loss to the players. However, many did not recognise that this loss to the players represented an expected return to the casino of $100.

QUESTION 17:

Parts (a) and (b) were well done. In part (c), a significant number of candidates created the new confidence interval correctly and then stated that the confidence interval had halved.

QUESTION 18:

A common error was to "estimate" the required probabilities. For example, in part (b) candidates needed the probability statement that equals 0.69. One standard deviation either side of the mean is 90 → 110: i.e. $P(90 < X < 110) = 0.68$. Candidates used this as their answer, or estimated $P(90 < X < 111) = 0.69$.

It was expected that students would use their understanding of the normal curve to write accurate statements, not estimates.

PART 2 CALCULATORS PERMITTED

More work needs to be done to improve effective use of CAS calculators. Settings were often incorrect, especially in relation to degrees and radians. Candidates should not be adding "calculator commands" as part of any solution. When "algebraic reasoning" is called for, students need to carefully consider the type and level of working required.

PART 2 SECTION A FUNCTION STUDY (C4)

QUESTION 19

Generally well done. Some candidates had difficulty using the formula $P = \frac{k}{V}$ to find *k*, and a few were unable to find an average. There were accuracy issues with calculations and a number of candidates had answers that were "illogical" in the context of the question, (i.e. negative volumes).

QUESTION 20

a) Well done.

b) Generally well done. However, a number of candidates did not read the instructions to include the asymptote, y-intercept and the point at *x*=3. Most graphs were well drawn.

c) Most candidates knew how to use algebra to find the inverse function, and how to sketch the inverse graph from *f(x)*. However, there was a lack of care with the sketch of the inverse. Given a labelled grid was supplied, markers expected the inverse graph to be correctly located on the axes. Generally candidates who added the line $y = x$ were the most successful here.

d) Well done. Although, there were a number of candidates who confused the required transformations with finding the inverse of $f(x)$.

e) Generally well done.

QUESTION 21

a) Poorly done. Most did not understand how a dilation by a factor of ½ in the

x direction would affect the graph. A common error was to place the curve at (4, 0) instead of (2, 0). Another was to dilate by a factor of 2 instead of by ½.

b) Poorly done. Many were able to do the initial substitution, and get $[2(2x) - 8]^2$ or $(4x - 8)^2$. Very few could express the answer in the required form of $a(x - h)^2$.

c) A number of candidates were unsure how to do this question. Also, errors made in part b meant few could do the required substitution successfully.

PART 2 SECTION B TRIGONOMETRY (C5)

QUESTION 22

Most candidates recognised $\frac{\pi}{4}$ as basic angle and either used symmetry or general formula.

Main problems with showing full algebraic working and reverting to CAS early.

Problems recognising required domain, not excluding $\frac{49\pi}{24}$ or finding $-\frac{\pi}{4}$.

Candidates who chose to transform the domain to the argument of the cos function, often obtained $[0, 4\pi]$.

Some problems transforming with brackets with most common mistake to subtract $\frac{\pi}{6}$ instead of adding.

QUESTION 23

a) Done well

b) Done poorly with most common error combining radians with degrees given.

c) Done well with most candidates getting $\pm 30\ or\ \pm\frac{\pi}{6}$ or multiples of these values.

d) Most candidates realised a substitution was necessary. Often the (90,1) was used and this does not help determine *a*. Often blending of radians and degrees in equations setup and additional confusion relating to CAS settings as a result. A common answer of a=53.86 indicated that the calculator was set inappropriately. Advice being that candidates work in one angle measure only. Some candidates got negative values for *a* which conflicted with the shape of the graph.

QUESTION 24

a) Done well. As grid was provided, expecting where curve intersected points that sketch was drawn carefully. Some ignored upper domain restriction. Some drew sketch as a small gradient line, indicating calculator was set in degrees.

b) Most candidates had difficulty understanding the meaning of this problem. While the time interval could have been interpreted in a variety of ways, many incorrectly substituted t=1. Many candidates omitted explanations which made it hard to award part marks. Answers using t=2.5 to 3.5, or t=2 to 3, or t=3 to 4 were the common valid responses.

c) Main problem was lack of explanations linking to solutions. This question was poorly answered due to interpretation issues. Credit given to candidates recognising translation correctly although unable to evaluate *c* correctly due to missing of expansion of brackets. This led to common answer of c=-1.5

PART 2 SECTION C DIFFERENTIAL CALCULUS (C6)

QUESTION 25

a) Well done by most candidates. The few errors that occurred were arithmetic and attempting to solve $\frac{dy}{dx} = 3$ rather than substituting $x = 3$ into $\frac{dy}{dx}$.

b) Reasonably well done by most candidates. Commonly candidates (correctly) stated that $\frac{dy}{dx} = 0$ had no solutions without explaining why. Correct justification included a graph of $\frac{dy}{dx}$ versus *x* (that doesn't intersect the *x* axis) or showing that $\Delta < 0$ in the quadratic equation that $\frac{dy}{dx} = 0$ gives.

QUESTION 26

There was a large range of responses for this question.

Often the average rate of change in (b) was interchanged with the instantaneous rate of change (a) – indicating conceptual shortcomings. Also the (rate) units of $/year were frequently omitted or only partially included.

a) Some candidates (refreshingly few) erroneously attempted to use $\frac{dx^n}{dx} = nx^{n-1}$ to differentiate the exponential function.

b) A frequent error was to use P' in the average rate of change, and so finding the average rate of change of P' not P.

Another common error was assuming the exponential term $1000e^{-0.5t}$ in $P(t)$ evaluates to 0 when $t = 0$ or evaluating $P(t)$ starting at 1

QUESTION 27

Overall the question was when understood and completed by candidates.

Shortcomings observed were:

Omitting the y coordinates of the stationary points.

Omitting the gradient indicators (or sign of $f'(x)$) in a table justifying/finding the nature of the SPs.

Occasionally, in evaluating $f(0) = 0^2(0-2) = -2$ was seen.

The second derivative was often successfully used to justify/find the nature of the SPs.

A graph of the gradient function (an efficient approach to justify/find the nature of the SPs.) was rarely seen.

QUESTION 28

Overall candidates had wholehearted attempts (at what is a more challenging application question).

a) The result was demonstrated successfully by many (others left it but used as intended the result to make progress in part b)
b) Some candidates' solutions did not fully show the formation of the formula, omitting steps in their proofs.

An error frequently seen was to take the (partial) derivative of $V = \frac{4}{3}\pi r^3 + \pi r^2 l$ with respect to r ($\frac{\partial V}{\partial r} = 4\pi r^2 + 2\pi r l$) and then substitute $l = -2r + \frac{7}{2\pi r}$. This is not the correct value of $\frac{dV}{dr} = -2\pi r^2 + \frac{7}{2}$ which is obtained by substituting $l = -2r + \frac{7}{2\pi r}$ then differentiating with respect to r. (In the former approach l a function of r is never differentiated).

Other errors were omitting explicit rejection of negative r solution to $\frac{dV}{dr} = 0$ and failing to justify why the stationary point corresponding to the positive r solution to $\frac{dV}{dr} = 0$ gives a maximum V.

PART 2 SECTION D INTEGRAL CALCULUS (C7)

QUESTION 29

Poorly handled by many candidates who didn't realise the need to split the integral into

$\int_1^4 \frac{2f(x)}{3}dx + \int_1^4 \frac{1}{3}dx.$

Those that did, generally handled the definite integral calculations well giving 5 as correct solution.

Many didn't tackle this problem which was surprising given that this type of question is common in past papers.

QUESTION 30

Most candidates correctly integrated the velocity equation to find the displacement. Many omitted the constant and subsequent evaluation of $c = 0$ using $t = 0$ and $s = 0$. Only part marks were allocated for a definite integral calculation using $s(t) = \int_0^{10} v(t)dt$ as this strictly finds the change in displacement and this only coincidentally gives correct answer as $s(0)=0$. Most candidates correctly handled the substitution of $t = 10$ to determine 30m. Some candidates had their calculators set in degrees which led to $s = 38.11$m

QUESTION 31

Poorly handled. Although a diagram was not essential, it should have provided clarity regards the two distinct areas above and below the x axis requiring two separate integrals. Candidates used the absolute function, a negative or bound swap to correctly deal with the area below the x axis. However, many of the diagrams were incorrect with the exponential function not passing through the origin. The most common error was $A = \int_{-ln3}^{ln3} e^x - 1\, dx$. The x bounds were often sketched as $y = -\ln 3$ and $y = \ln 3$ which added to the integration confusion. The dx was often omitted from the definite integrals. The integration was handled well but there were many substitution and simplification errors.

QUESTION 32

Very few candidates received full marks for this question as little regard was given to the "full algebraic working" instruction. Most candidates recognised the $A = 2\int_0^{10} 2\sqrt{\frac{2x}{5}}\,dx$ using symmetry.

Once stating the definite integral, most candidates then used CAS to generate an integral answer like $\left[\frac{4\sqrt{10}(x)^{3/2}}{15}\right]$ instead of something like $\left[4(\frac{2x}{5})^{3/2}.\frac{2}{3}.\frac{5}{2}\right]$ which showed a level of algebraic detail required.

A common error was $A = \int_0^{20} 2\sqrt{\frac{2x}{5}}\,dx$. Some candidates wasted valuable time working out the equation for the reflected half but few correctly obtained

$$A = \int_0^{10} 2\sqrt{\frac{2x}{5}}\,dx + \int_{10}^{20} 2\sqrt{\frac{40-2x}{5}}\,dx$$

The calculating of a volume using $V = A \times l$ was handled well. Units were included correctly in most cases.

QUESTION 33

Many candidates didn't recognise the tangent had a gradient of 4 which gives $k = \sqrt{2}$ after solving $4 = 2 + k\sqrt{2}$. Once finding k, many integrated but could not determine a way of finding the constant. Realising that at $x = 2$, $y = 5$ and that this point lies on the tangent was often not understood. Hence, many found it difficult to successfully work their way into the problem. Those who handled these steps generally used the (2, 5) to determine c as a third and stating the function as:

$$f(x) = \frac{x^2}{2} + \frac{2\sqrt{2}\,\sqrt{x^3}}{3} + \frac{1}{3}$$

PART 2 SECTION E PROBABILITY (C8)

Students must define both the variable and distributions. Calculator commands in this section are not relevant. (omit responses like "binomcdf(…..)"

QUESTION 34

This straightforward question was well answered by many candidates. However, there were a number who made heavy weather in obtaining their results, often not correctly evaluating the probability of success.

A common error was to add probabilities for 'and' results rather than multiplying.
A number of candidates who gave answers only did this to their detriment: a brief (correct) numerical statement would have yielded some result, rather than nothing.

QUESTION 35

This was generally well done. Of those candidates who understood the method required, errors were involved by unnecessarily rounding the appropriate *z*-score. Furthermore, a number of candidates neglected to include units in their answer.

QUESTION 36

a) Most candidates understood what was required to answer this question. However, there was a number who were unable to gain full marks by simply stating the result but giving no working. Other issues involved unnecessarily rounding answers and not recognising the "at least" condition.

b) Those candidates who recognised $n = 10$ and used the result for p from part (a) were successful. Once again, marks were lost for simply writing the answer.

QUESTION 37

a) Generally well done.

b) A number of candidates set up $\hat{p}$ and $(1 - \hat{p})$ but had trouble in setting M correctly. Furthermore, transformation of the formula to find n was also a significant stumbling block.

c) Few candidates were able to get far with this question. A major error occurred when re-evaluating $\hat{p}$ and $(1 - \hat{p})$ based on the 51%. A significant number of those who attempted it recognised the need to use "n" from part (b), but few recognised the new margin of error to be 0.035.

POSSIBLE SOLUTIONS

PART 1

SECTION A

QUESTION 1

$(ax + b)^3$ is equal to $8x^3 - 12x^2 + 6x - 1$. Find the values of a and b. (2 marks)

The x^3 coefficient is $(ax)^3 = a^3x^3 = 8x^3$ so $a^3 = 8$ thus $a = 2$.

The constant coefficient is $b^3 = -1$ thus $b = -1$.

$\therefore \; a = 2, b = -1$.

QUESTION 2

A polynomial is given by $(x) = ax^4 - 2x^3 + bx - 3$, where $a, b \in \mathbb{R}$. When $P(x)$ is divided by $(x + 1)$ the remainder is 5.

When $P(x)$ is divided by $(x - 1)$ the remainder is -3.

Find $\boldsymbol{a}$ and $\boldsymbol{b}$.

(3 marks)

$P(-1) = 5$ so $a + 2 - b - 3 = 5$.

$\therefore \; a - b = 6$ - Eq. 1

$P(1) = -3$ so $a - 2 + b - 3 = -3$.

$\therefore \; a + b = 2$ - Eq. 2

Eq. 1 + Eq. 2 $2a = 8$

$\therefore \; a = 4$.

$\therefore \; b = -2$.

QUESTION 3

Find the zeroes of the truncus function $\boldsymbol{f(x) = \frac{1}{(x-6)^2} - 4}$, $\boldsymbol{x \neq 6}$.

(3 marks)

Sub. $f(x) = 0$.

$$\therefore \frac{1}{(x-6)^2} - 4 = 0$$

$$\therefore \frac{1}{(x-6)^2} = 4$$

$$\therefore (x-6)^2 = \frac{1}{4}$$

$$\therefore x - 6 = 2 \pm \frac{1}{2}$$

$\therefore \ x = 5\frac{1}{2}$ or $6\frac{1}{2}$.

QUESTION 4

Solve $2 \times 16^x + 3 \times 4^x = 2$ using the substitution $A = 4^x$.

(4 marks)

We have $2 \times A^2 + 3 \times A = 2$

$$\therefore 2A^2 + 3A - 2 = 0$$

$$\therefore (2A-1)(A+2) = 0$$

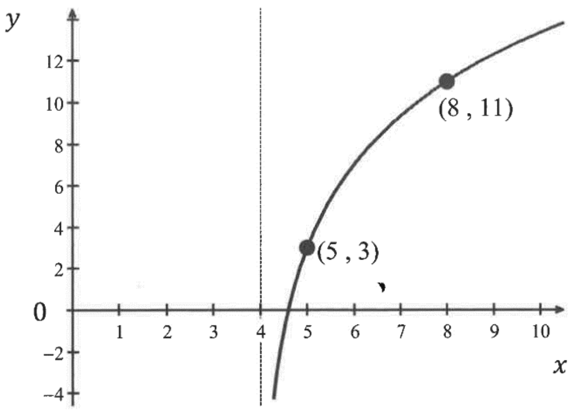
$\therefore \ A = \frac{1}{2}$ or $A = -2$

$$\therefore 4^x = \frac{1}{2} \ or \ 4^x = -2 \ (no \ valid \ solution)$$

$$x = \log_4 \frac{1}{2} = \log_4 2^{-1} = -\log_4 2 = -\frac{1}{2}.$$

QUESTION 5

A logarithmic equation has the form $y = log_2(x - h) + k$ and is pictured below.



(a) What is the domain and range of the logarithmic equation?

(1 mark)

Domain: $(4, \infty)$

Range: $\mathbb{R}$

(b) Find the values of a, h and k.

(3 marks)

The VA gives that $h = 4$.

Sub. $x = 5$, $y = 3$.

$\therefore\ 3 = a \times \log_2(5 - 4) + k$

$\therefore\ 3 = a \times 0 + k$

$\therefore\ k = 3.$

Sub. $x = 8$, $y = 11$.

$\therefore\ 11 = a \times \log_2(8 - 4) + 3$

$\therefore\ 8 = a \times 2$

$\therefore\ a = 4.$

$\therefore\ a = 4,\ h = 4,\ k = 3.$

SECTION B

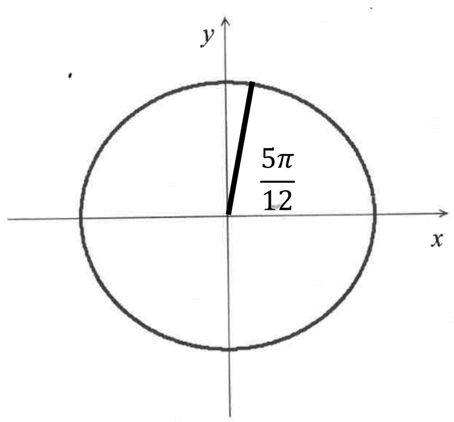
QUESTION 6

(a) Convert $\frac{5\pi}{12}$ radians into degrees.

(1 mark)

$$= \frac{5\pi}{12} \times \frac{180}{\pi} = 5 \times 15 = 75°$$

(b) Identify on the diagram where an angle of $\frac{5\pi}{12}$ will be.

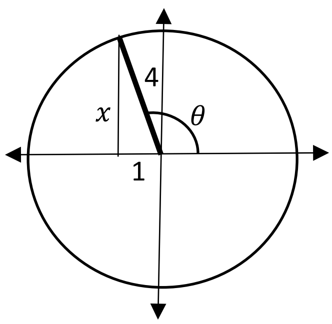


(1 mark)

QUESTION 7

Given that $\boldsymbol{cos\theta} = -\frac{1}{4}$ and $\frac{\pi}{2} < \boldsymbol{\theta} < \boldsymbol{\pi}$, evaluate $\boldsymbol{sin\theta}$ and $\boldsymbol{tan\theta}$.

(3 marks)



$$x = \sqrt{4^2 - 1^2} = \sqrt{15}$$

$$\therefore \quad sin\theta = +\frac{\sqrt{15}}{4}$$

$$\therefore \quad tan\theta = -\sqrt{15}.$$

OR $cos^2 + sin^2 = 1$ $\quad \left(\frac{-1}{4}\right)^2 + \sin\theta^2 = 1$ $\quad \tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad \tan\theta = \frac{\sqrt{15}/4}{-1/4}$

$\sin\theta^2 = 1 - \frac{1}{16}$ $\quad \tan\theta = -\sqrt{15}$

$\sin\theta = \frac{\pm\sqrt{15}}{4}$ [sin +'ve in 2nd quad] $\quad \therefore \sin\theta = \frac{\sqrt{15}}{4}$, $\tan\theta = -\sqrt{15}$

QUESTION 8

Solve $\boldsymbol{tan}\left(\boldsymbol{x}-\frac{\boldsymbol{\pi}}{\boldsymbol{4}}\right)=\mathbf{1}$ for $\boldsymbol{x}\in[\boldsymbol{\pi},\boldsymbol{3\pi}]$

(3 marks)

$$tan\theta = 1\ for\ \theta = \frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\frac{13\pi}{4},\dots\dots..$$

$$\therefore\ x-\frac{\pi}{4}=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\frac{13\pi}{4},\dots\dots..$$

$$\therefore\ x=\frac{2\pi}{4},\frac{6\pi}{4},\frac{10\pi}{4},\frac{14\pi}{4},\dots\dots..$$

$$\therefore\ x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\dots\dots..$$

$$Reject\ x=\frac{\pi}{2} and\ \frac{7\pi}{2}\ as\ outside\ domain$$

$$\therefore\ x=\frac{3\pi}{2},\frac{5\pi}{2}$$

OR

$$x-\frac{\pi}{4}=n\pi+artan(1)$$

$$\therefore\ \ x=n\pi+\frac{\pi}{4}+\frac{\pi}{4}=n\pi+\frac{\pi}{2}.$$

$n=1$: $x=\frac{3\pi}{2}$ Yes.

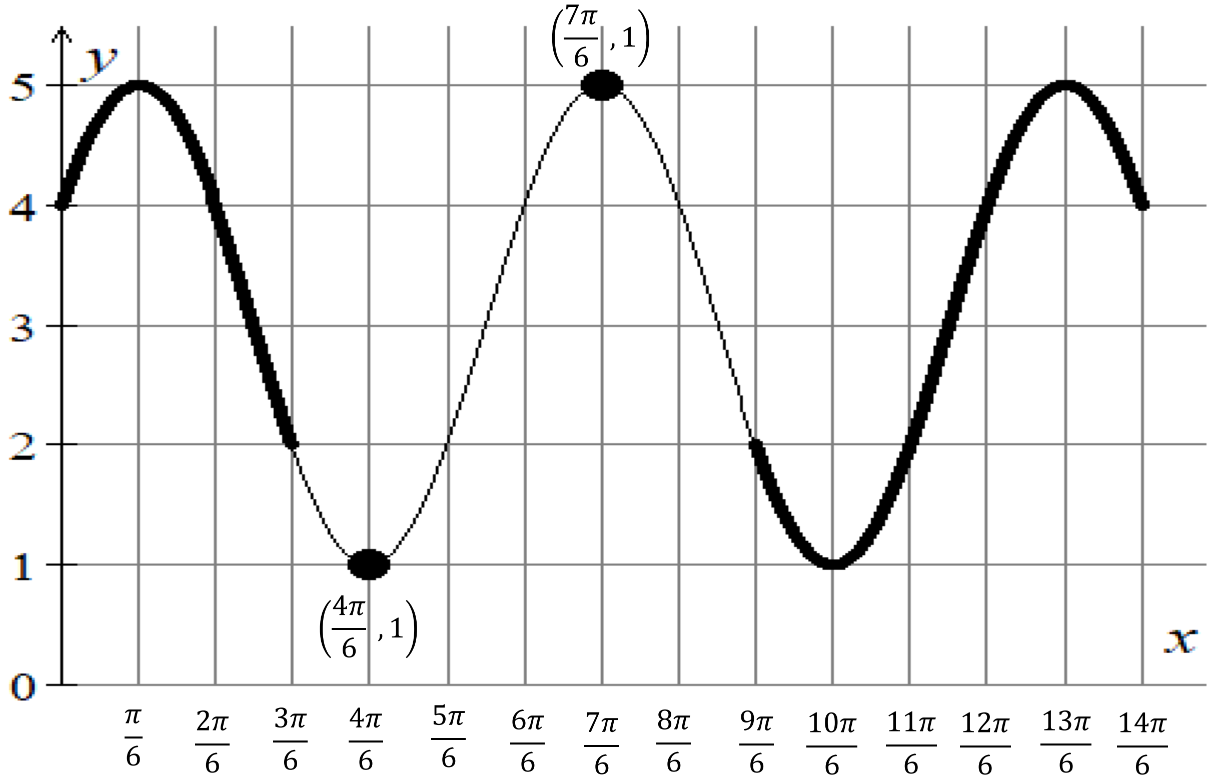
$n=2$: $x=\frac{5\pi}{2}$ Yes.

$n=3$: $x=\frac{7\pi}{2}$ No.

$\therefore\ x=\frac{3\pi}{2},\frac{5\pi}{2}$

QUESTION 9

Below is a partly drawn graph from an equation of the form $y = a\cos n(x+b)+c$



(a) Complete the sketch of the function, labelling the missing minimum and maximum points.

(3 marks)

(b) Find the values of:

(i) $a = 2$ (1 mark)

(ii) $c = 3$ (1 mark)

(iii) period $= \frac{6\pi}{6} = \pi$ so $n = \frac{2\pi}{\pi} = 2$ (1 mark)

(iv) $b = -\frac{\pi}{6}$ (translated $\frac{\pi}{6}$ to the right) [Note: other possibilities] (1 mark)

(c) Check your answers to (b) by calculating y when $x = \frac{\pi}{6}$.

$$y = cos\left(2\left(\frac{\pi}{6} - \frac{\pi}{6}\right)\right) + 3$$

$$= 2cos2(0) + 3$$

$$= 2cos0 + 3$$

$$= 5$$

SECTION C

QUESTION 10

Evaluate the following limits

(a) $\lim_{x \to -2} \frac{3x+1}{2x-5}$

(1 mark)

$$= \frac{3 \times -2 + 1}{2 \times -2 - 5}$$

$$= \frac{-6 + 1}{-4 - 5}$$

$$= \frac{5}{9}$$

(b) $\lim_{x \to 7} \frac{(x-7)^2}{(3x-7)(x-7)}$

(2 marks)

$$\lim_{x \to 7} \frac{x - 7}{3x - 7}$$

$$= \frac{0}{14}$$

$$= 0.$$

QUESTION 11

Determine the derivatives of (no simplification required):

(a) $f(x) = 3x^2 sin(x)$

(2 marks)

$f'(x) = 6xsin(x) + 3x^2 cos(x)$

(b) $y = e^{4ln(2x)}$

(2 marks)

$y = (e^{ln2x})^4$

$= (2x)^4$

$= 16x^4$

So $\frac{dy}{dx} = 64x^3$.

OR

$\frac{dy}{dx} = e^{4ln(2x)} \times \frac{4}{2x} \times 2$

$= \frac{4}{x} e^{4ln(2x)}$

(c) $g(x) = \frac{(4-x)}{(2x+1)^2}$

(2 marks)

$$g'(x) = \frac{-(2x+1)^2 - (4-x)2(2x+1) \times 2}{(2x+1)^4}$$

$$= \frac{-(2x+1)^2 - 4(4-x)(2x+1)}{(2x+1)^4} = \frac{-(2x+1) - 4(4-x)}{(2x+1)^3} = \frac{2x-17}{(2x+1)^3}$$

OR

$g(x) = (4-x)(2x+1)^{-2}$

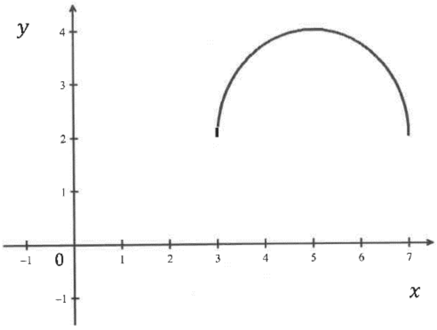
$\therefore\ g'^{(x)} = -1 \times (2x+1)^{-2} + (4-x) \times -2(2x+1)^{-3} \times 2$

$= -(2x+1)^{-2} - 4(4-x)(2x+1)^{-3}$

QUESTION 12

The curve shown on the graph can be represented by the function:

$$\boldsymbol{y = \sqrt{4 - (x - 5)^2} + 2}.$$



(a) Find the gradient of the tangent to the curve at $\boldsymbol{x = 6}$.

(3 marks)

$$\frac{dy}{dx} = \frac{1}{2}(4 - (x - 5)^2)^{-\frac{1}{2}} \times -2(x - 5) \times 1$$

$$= \frac{x - 5}{-(4 - (x - 5)^2)^{\frac{1}{2}}}$$

When $x = 6$

$$\frac{dy}{dx} = \frac{1}{-(4 - 1)^{\frac{1}{2}}} = -\frac{1}{\sqrt{3}}$$

(b) Hence show that the equation of the normal to the curve at $x = 6$ is given by $y = \sqrt{3}x - 5\sqrt{3} + 2$.

(3 marks)

Slope of normal $= -\frac{1}{-\frac{1}{\sqrt{3}}} = \sqrt{3}$.

When $x = 6$, $y = \sqrt{4 - (6 - 5)^2} + 2 = \sqrt{3} + 2$.

So $y - (\sqrt{3} + 3) = \sqrt{3}(x - 6)$

$\therefore \quad y = \sqrt{3}x - 5\sqrt{3} + 2$.

(c) Show that the normal passes through the point $(5, 2)$.

(1 mark)

When $x = 5$:

$y = \sqrt{3} \times 5 - 5 \times \sqrt{3} + 2 = 2$.

So the normal passes through $(5, 2)$.

SECTION D

QUESTION 13

Evaluate the following integrals:

(a) $\int \frac{cos(3x+1)}{2} dx$

(2 marks)

$$= \frac{1}{2} \times \frac{sin(3x+1)}{3} + C$$

$$= \frac{sin(3x+1)}{6} + C$$

(b) $\int_1^3 \left(3x^2 + 7x + \frac{3}{x}\right) dx$

(3 marks)

$$= \left[x^3 + \frac{7x^2}{2} + 3ln|x|\right]_1^3$$

$$= 3^3 + \frac{7 \times 3^2}{2} + 3ln3 - 1^3 - \frac{7 \times 1^2}{2} - 3ln1$$

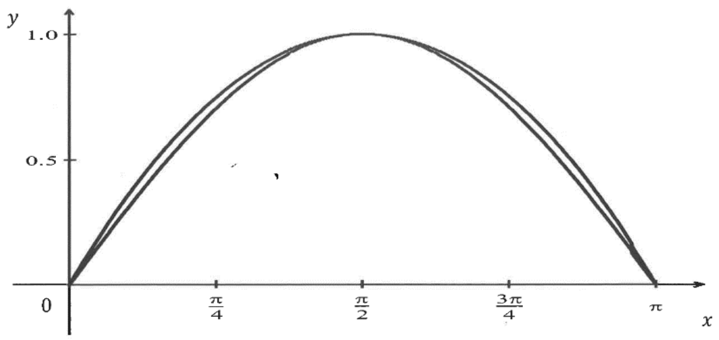
$$= 27 + \frac{63}{2} + 3ln3 - 1^3 - \frac{7}{2}$$

$$= 3ln3 + 54.$$

QUESTION 14

The graph of $f(x) = sin(x)$ can be modelled very closely by the parabola

$g(x) = \frac{4}{\pi^2} x(\pi - x)$ over the interval $[0, \pi]$ as shown on the graph below.



(a) Show that $g\left(\frac{\pi}{4}\right) = \frac{3}{4}$.

(1 mark)

$g\left(\frac{\pi}{4}\right) = \frac{4}{\pi^2} \times \frac{\pi}{4}\left(\pi - \frac{\pi}{4}\right)$

$= \frac{1}{\pi} \times \frac{3\pi}{4} = \frac{3}{4}.$

(b) Since $\frac{3}{4} > sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ then $g(x)$ is the uppermost function.

Find the exact area between the two functions over the interval $[0\,,\pi]$.

(4 marks)

$$= \int_0^\pi \left(\frac{4}{\pi^2}x(\pi - x) - sinx\right) dx$$

$$= \int_0^\pi \left(\frac{4}{\pi}x - \frac{4}{\pi^2}x^2 - sinx\right) dx$$

$$= \left[\frac{4}{\pi} \times \frac{x^2}{2} - \frac{4}{\pi^2} \times \frac{x^3}{3} + cosx\right]_0^\pi$$

$$= \left[\frac{2x^2}{\pi} - \frac{4x^3}{3\pi^2} + cosx\right]_0^\pi$$

$$= 2\pi - \frac{4\pi}{3} + (-1) - (0 - 0 + 1)$$

$$= \frac{2\pi}{3} - 2\;units^2.$$

QUESTION 15

(a) Show that the derivative of $\frac{xlnx - x}{ln2}$ is equal to $\frac{lnx}{ln2}$.

(2 marks)

$$\frac{d}{dx}\left(\frac{xlnx - x}{ln2}\right)$$

$$= \frac{1 \times lnx + x \times \frac{1}{x} - 1}{ln2}$$

$$= \frac{lnx + 1 - 1}{ln2}$$

$$= \frac{lnx}{ln2}$$

(b) Using your results in part (a) show that

(4 marks)

$$\int_2^4 \left(\frac{lnx}{ln2} - 3\right) = \left[\frac{xlnx - x}{ln2} - 3x\right]_2^4$$

$$= \frac{4ln4 - 4}{ln2} - 12 - \left(\frac{2ln2 - 2}{ln2} - 6\right)$$

$$= 4 \times 2 - \frac{4}{ln2} - 12 - 2 + \frac{2}{ln2} + 6$$

$$= 8 - 12 - 2 + 6 - \frac{2}{ln2}$$

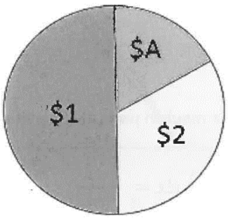
$$= -\frac{2}{ln2}.$$

SECTION E

QUESTION 16

In a casino game, $\boldsymbol{X}$ is the winnings paid out when the wheel pictured opposite is spun. The probabilities of each payout are shown below:

x	1	2	A
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	k



(a) Find the value of $\boldsymbol{k}$.

(1 mark)

$$\frac{3}{6} + \frac{2}{6} + k = 1$$

$$\therefore \ k = \frac{1}{6}.$$

(b) Given that the expected payout, $E(X)$, is to be \$2, find the value of A.

(2 marks)

$$E(X) = 1 \times \frac{1}{2} + 2 \times \frac{1}{3} + A \times \frac{1}{6}$$

$$\therefore \ 2 = \frac{3}{6} + \frac{4}{6} + \frac{A}{6}$$

$$\therefore \ A = 5.$$

(c) Find the variance of the payout, $var(X)$.

(3 marks)

$var(X) = E(X^2) - E(X)^2$

$= 1 \times \frac{1}{2} + 4 \times \frac{1}{3} + 25 \times \frac{1}{6} - 2^2$

$= \frac{3}{6} + \frac{8}{6} + \frac{25}{6} - 4$

$= 6 - 4 = 2.$

(d) The casino is told that the expected return of the game from 100 plays will be $E(100X - 300)$.

(2 marks)

$= E(100X - 300) = 100 \times E(X) - 300$

$= 100 \times 2 - 300$

$= -100$

So the casino makes a profit of $100.

QUESTION 17

In a phone poll of 1000 people it is found that 400 answer "yes" to a survey question.

(a) What is the proportion in the sample, $\hat{\boldsymbol{p}}$, who would answer "yes" to the survey question?

(1 mark)

$\hat{p} = \frac{400}{1000} = 0.4$

(b) Write an expression for the 95% confidence interval for the proportion of people who answered "yes".

(2 marks)

$$= \left(0.4 - 1.96 \times \sqrt{\frac{0.4 \times 0.6}{1000}}, 0.4 + 1.96 \times \sqrt{\frac{0.4 \times 0.6}{1000}}\right)$$

(c) By what factor will the margin of error for the 95% confidence interval change if 2000 people were sampled instead, and 800 answered "yes".

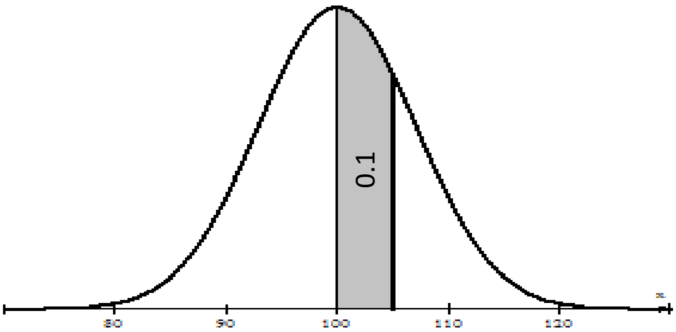
(2 marks)

$\hat{p}$ will stay the same, but n will double so the interval will change by a factor of $\frac{1}{\sqrt{2}}$.

QUESTION 18

X is a normal distribution with mean 100 and standard deviation 10.

For the shaded area above the following probability statement can be made:



$$\boldsymbol{P(100 \leq X \leq 105) = 0.19}$$

Write probability statements for this distribution that represent:

(a) 0.31

(1 mark)

$P(X \geq 105)$ *or* $P(X > 105)$

(b) 0.69

(1 mark)

$P(X \leq 105)$ *or* $P(X < 105)$

(c) 0.38

(1 mark)

$P(95 \leq X \leq 105)$ *or* $P(95 < X < 105)$

PART 2

SECTION A

QUESTION 19

A science student is studying the relationship between the pressure and volume of a gas. The following is a sample of the student's results:

Pressure P (mm Hg)	Volume V (mL)
944	24.9
890	26.4

(a) The data can be modelled in the form $\boldsymbol{P} = \frac{\boldsymbol{k}}{\boldsymbol{V}}$ where $\boldsymbol{k}$ is a constant.

Using this data, find the average value of $\boldsymbol{k}$.

(2 marks)

$k = V \times P = 24.9 \times 944 = 23505.6$

also
$k = V \times P = 26.4 \times 890 = 23496$

The average value of k is 23500.8.

(b) Predict the volume when the pressure of the gas is 500 mm Hg.

(2 marks)

When $P = 500$.

$V = \frac{k}{P} = \frac{23500.8}{500} = 47.0$ mL.

QUESTION 20

(a) The function $y = e^x$ is translated right by 3 units and up by 1 unit to give the Function $f(x)$. Write down the equation of $f(x)$.

(1 mark)

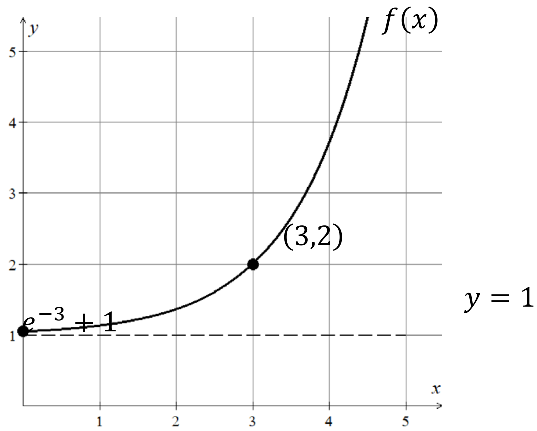
$f(x) = e^{x-3} + 1$.

(b) Sketch the graph of $f(x)$ on the axes below.

Include the asymptote and the y-intercept.

Label the point where $x = 3$.

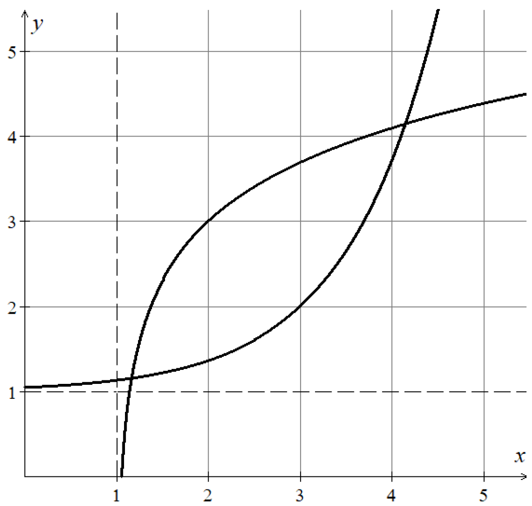
(3 marks)



(c) Sketch the inverse on the same set of axes above.

Algebraically find the inverse function $f^{-1}(x)$.

(3 marks)



Let $y = e^{x-3} + 1$.

Swap x and y.

$\therefore \; x = e^{y-3} + 1$

$\therefore \; ln|x - 1| = y - 3$

$\therefore y = ln|x-1| + 3$

$\therefore \ f^{-1}(x) = ln|x-1| + 3.$

(d) State the transformations required to transform the equation

$y = ln(x)$ into $f^{-1}(x)$.

(1 mark)

The graph of $ln|x|$ is translated 1 right and 3 up.

(e) An airplane wing is to be made from the area between these two curves. State the restricted domain of $f(x)$ that forms the wing.

Give your answer to 2 decimal places.

(1 mark)

Solve $f(x) = f^{-1}(x)$.

$\therefore \ x = 1.159$ or $x = 4.146$.

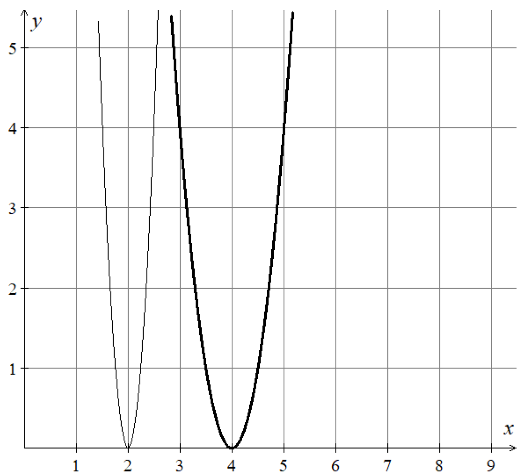
So the required domain is $(1.16, 4.15)$ or $[1.16, 4.15]$ $depending$ on $interpretation$.

QUESTION 21

The sketch of $f(x) = (2x-8)^2$ is given below.

(a) On the same axes, sketch the graph after a dilation of factor $\frac{1}{2}$ in the x-direction has been applied.

(2 marks)



(b) The formula sheet says that a dilation in the direction of the x-axis by a factor of $\frac{1}{2}$ is $f(x) \rightarrow f(2x)$.

This dilation can also be determined using the function $g(x) = 2x$ and finding $f\big(g(x)\big)$. Find $f\big(g(x)\big)$ and express it in the form $y = a(x-h)^2$.

(3 marks)

$f\big(g(x)\big) = f(2x) = (2(2x) - 8)^2$

$= (4x - 8)^2$

$= 4^2 \times (4x - 8)^2$

$= 16(x - 2)^2$

(c) Using your answer to (b) show that the point on the original graph $(5, 4)$ correctly maps to the dilated point $\left(\frac{5}{2}, 4\right)$.

(2 marks)

$f\left(g\left(\frac{5}{2}\right)\right) = 16\left(\frac{5}{2} - 2\right)^2$

$= 16 \times \left(\frac{1}{2}\right)^2 = 4.$

SECTION B

QUESTION 22

Solve $2cos\left(2\left(x-\frac{\pi}{6}\right)\right)=\sqrt{2}$, for $0\le x\le 2\pi$.

Show all algebraic working in your answer.

(6 marks)

$$Basic\ angle\ \frac{\pi}{4}\ and\ \cos +\ in\ 1^{st}\ and\ 4^{th}\ quadrants$$

$$\therefore\ 2\left(x-\frac{\pi}{6}\right)=\frac{\pi}{4},2\pi-\frac{\pi}{4},2\pi+\frac{\pi}{4},4\pi-\frac{\pi}{4},-\frac{\pi}{4},\ldots..$$

$$\therefore\ \left(x-\frac{\pi}{6}\right)=\frac{\pi}{8},\frac{7\pi}{8},\frac{9\pi}{8},\frac{15\pi}{8},-\frac{\pi}{8}\ldots..$$

$$\therefore\ \left(x-\frac{4\pi}{24}\right)=\frac{3\pi}{24},\frac{21\pi}{24},\frac{27\pi}{24},\frac{45\pi}{24},-\frac{3\pi}{24}\ldots..$$

$$\therefore\ x=\frac{7\pi}{24},\frac{25\pi}{24},\frac{31\pi}{24},\frac{49\pi}{24},\frac{\pi}{24}$$

$$\therefore\ x=\frac{\pi}{24},\frac{7\pi}{24},\frac{25\pi}{24},\frac{31\pi}{24}\ after\ rearranging\ and\ considering\ domain$$

OR

$$cos\left(2\left(x-\frac{\pi}{6}\right)\right)=\frac{\sqrt{2}}{2}$$

$$\therefore\ 2\left(x-\frac{\pi}{6}\right)=2n\pi\pm arcos\left(\frac{\sqrt{2}}{2}\right)$$

$$\therefore\ 2\left(x-\frac{\pi}{6}\right)=2n\pi\pm\frac{\pi}{4}$$

$$\therefore\ x-\frac{\pi}{6}=n\pi\pm\frac{\pi}{8}$$

$$\therefore\ x=n\pi\pm\frac{\pi}{8}+\frac{\pi}{6}$$

$n=0$: $x=\pm\frac{\pi}{8}+\frac{\pi}{6}=\frac{\pi}{24}$ and $\frac{7\pi}{24}$.

$n=1$: $x=\pi\pm\frac{\pi}{8}+\frac{\pi}{6}=\frac{25\pi}{24}$ and $\frac{31\pi}{24}$.

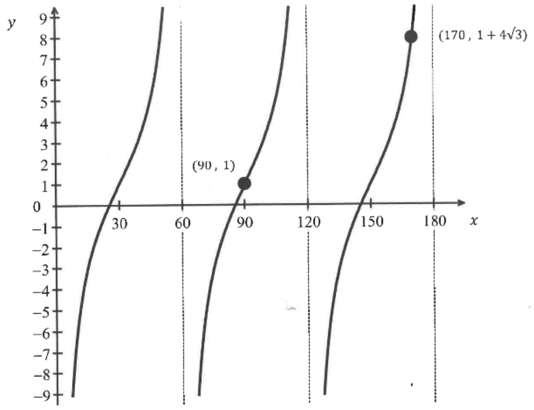
$n=2$: $x=2\pi\pm\frac{\pi}{8}+\frac{\pi}{6}$ No.

$$\therefore\ x=\frac{\pi}{24},\frac{7\pi}{24},\frac{25\pi}{24},\frac{31\pi}{24}.$$

QUESTION 23

The function below can be modelled by an equation of the form

$y = a\,tan\big(n(x + b)\big) + c$, where x is measured in degrees.



Find the values of a, n, b and c.

(a) $c = 1$

(1 mark)

(b) period = $60°$. $\therefore\; n = \frac{180}{60} = 3.$

(1 mark)

(c) $b = -30°$. Translated 30° right. [Note alternatives possible]

(1 mark)

(d) a

Sub. $x = 170$, $y = 1 + 4\sqrt{3}$.

$\therefore\; 1 + 4\sqrt{3} = a \times tan\big(3(170 - 30)\big) + 1$

$\therefore\; \frac{4\sqrt{3}}{a} = tan(3 \times 140) = \sqrt{3}$

$\therefore\; a = 4.$

(2 marks)

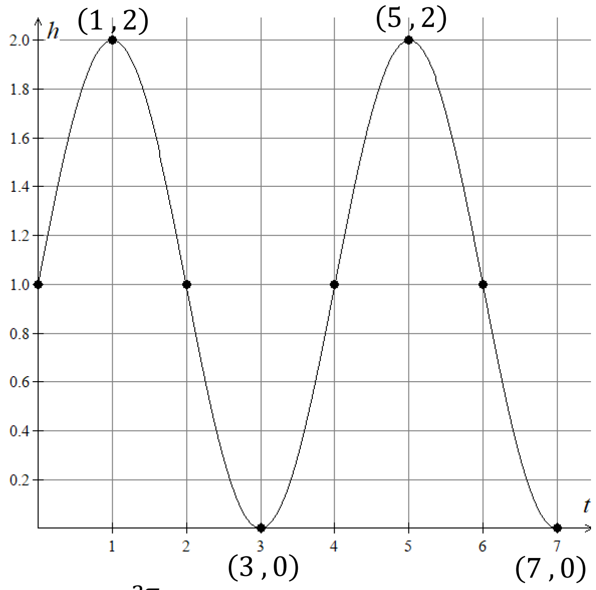
QUESTION 24

The height of the middle of a skipping rope can be modelled by the equation

$h = sin\left(\frac{\pi}{2}t\right) + 1$, for $0 \leq t \leq 7$ where t is in seconds and h is in metres.

(a) Sketch the graph of h for $0 \leq t \leq 7$.

(4 marks)



Period $= \frac{2\pi}{\pi/2} = 4$.

(b) A camera is set up to film the middle of the rope when it is at the bottom of its rotation. If the length of the filming is 1 second, find the value k for which

$sin\left(\frac{\pi}{2}t\right) + 1 \leq k$ during the filming.

(2 marks)

Find the height at 2.5 seconds.

$h = sin\left(\frac{\pi}{2} \times 2.5\right) + 1 = 0.2929$ or $h = 1 - \frac{\sqrt{2}}{2}$

So if $k = 0.2929$, the rope will be below k for the interval $(2.5 , 3.5)$.

(c) The start time is changed so that the height of the middle of the skipping rope is below k between 4 and 5 seconds.

Find a value of c so that $sin\left(\frac{\pi}{2}t + c\right) + 1 \leq k$ for $4 \leq t \leq 5$.

(3 marks)

We need to shift the graph 1.5 seconds to the right.

So $h = sin\left(\frac{\pi}{2}(t - 1.5)\right) + 1$.

$= sin\left(\frac{\pi}{2}t - \frac{3\pi}{4}\right) + 1$

$\therefore \quad c = -\frac{3\pi}{4}$.

SECTION C

QUESTION 25

The equation $y = x^3 - 9x^2 + 28x - 28$ is a cubic graph that does not have a stationary point.

(a) What is the value of the slope of the tangent at $x = 3$?

(2 marks)

$\frac{dy}{dx} = 3x^2 - 18x + 28$

At $x = 3$, $\frac{dy}{dx} = 3 \times 3^2 - 18 \times 3 + 28$.

$= 27 - 54 + 28$

$= 1$.

(b) Show the equation has no stationary points.

(2 marks)

Solve $\frac{dy}{dx} = 0$ to find stationary points.

$\therefore\ 3x^2 - 18x + 28 = 0$

Now $\Delta = (-18)^2 - 4 \times 3 + 28 = -12$.

$\therefore\ \frac{dy}{dx}$ has no real solutions.

$\therefore\ y = x^3 - 9x^2 + 28x - 28$ has no stationary points.

QUESTION 26

The price of a new computer can be modelled by the equation $P = 1000e^{-0{,}5t} + 200$, where P is the price ($) and t is the time in years.

(a) Calculate the rate of change in price after 5 years.

(2 marks)

$\frac{dP}{dt} = 1000 \times e^{-0.5t} \times -0.5$

$= -500e^{-0.5t}$.

When $t = 5$, $\frac{dP}{dt} = -500 \times e^{-2.5} = -41.04$ $/year.

(b) Calculate the average rate of change over the first 5 years.

(2 marks)

When $t = 0$, $P = 1000 + 200 = 1200$.

When $t = 5$, $P = 1000e^{-0.5\times 5} + 200 = 282.05$.

Average rate of change $= \frac{282.05-1200}{5-0} = -183.58$ \$/year.

QUESTION 27

Consider the function $f(x) = x^2(x^2 - 2)$.

Find and classify the stationary points for the function.

(5 marks)

$f'(x) = 2x(x^2 - 2) + x^2(2x)$

$= 2x^3 - 4x + 2x^3$

$= 4x^3 - 4x$

Solve $f'(x) = 0$ to find the stationary points

$\therefore \; 4x^3 - 4x = 0$

$\therefore \; 4x(x^2 - 1) = 0$

$\therefore \; x = \pm 1$ or $x = 0$.

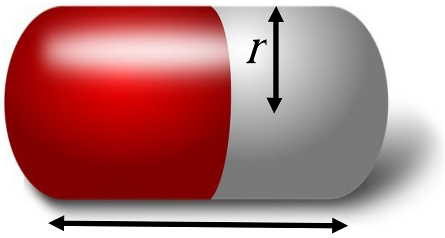
$f''^{(x)} = 12x^2 - 4$.

x	-1	0	1
$f(x)$	-1	0	-1
$f''(x)$	8	-4	8
	Min.	Max.	Min.

$\therefore \; (-1\,,-1)$ is a minimum, $(0\,,0)$ is a maximum, $(1\,,-1)$ is a minimum.

QUESTION 28

A medicine capsule is constructed by attaching the two halves of a sphere of radius r onto a cylinder with radius r and length ℓ as shown:



(a) The capsule is to have a total surface area (SA) of 7 cm^2.

Hence show that $\ell = -2r + \frac{7}{2\pi r}$.

(2 marks)

$SA = 4\pi r^2 + 2\pi r\ell = 7.$

$\therefore\ 2\pi r(2r + \ell) = 7$

$\therefore\ 2r + \ell = \dfrac{7}{2\pi r}$

$\therefore\ \ell = -2r + \dfrac{7}{2\pi r}$

(b) Using calculus and $V = \frac{4}{3}\pi r^3 + \pi r^2 \ell$ find the maximum volume that the capsule can hold with the given surface area.

(5 marks)

$V = \dfrac{4}{3}\pi r^3 + \pi r^2 \ell = \dfrac{4}{3}\pi r^3 + \pi r^2\left(-2r + \dfrac{7}{2\pi r}\right)$

$= \dfrac{4}{3}\pi r^3 - 2\pi r^3 + \dfrac{7r}{2}$

$= -\dfrac{2}{3}\pi r^3 + \dfrac{7r}{2}$

So $\frac{dV}{dr} = -2\pi r^2 + \frac{7}{2}$.

Solve $\frac{dV}{dr} = 0$ to find the stationary points.

$\therefore \ -2\pi r^2 + \frac{7}{2} = 0$

$\therefore \ r = -0.746$ (silly!) or $r = 0.746$

$\therefore \ r = \dfrac{\sqrt{7}}{2\sqrt{\pi}}$

$\frac{d^2V}{dr^2} = -4\pi r$ so when $r = \frac{\sqrt{7}}{2\sqrt{\pi}}, \frac{d^2V}{dr^2} < 0.$

$\therefore \ r = \frac{\sqrt{7}}{2\sqrt{\pi}}$ is the maximum value.

Maximum volume $= -\frac{2}{3}\pi\left(\frac{\sqrt{7}}{2\sqrt{\pi}}\right)^3 + \frac{7\sqrt{7}}{4\sqrt{\pi}}$

$= -\frac{\sqrt{7\pi}}{3} + \frac{7\sqrt{7}}{4\sqrt{\pi}} = \frac{7\sqrt{7}}{6\sqrt{\pi}} =$1.7415 cm^3.

Section D

Question 29

Given $\int_1^4 f(x)dx = 6$ find $\int_1^4 \frac{2f(x)+1}{3}dx$.

(3 marks)

$$\int_1^4 \frac{2f(x)+1}{3}dx = \frac{1}{3}\int_1^4 (2f(x)+1)dx$$

$$= \frac{2}{3}\int_1^4 f(x)dx + \frac{1}{3}\int_1^4 1dx$$

$= \frac{2}{3} \times 6 + \frac{1}{3}[x]_1^4 = 4 + 1 = 5.$

QUESTION 30

The velocity of a particle in m/s is given by $v(t) = 3 + cos(2\pi t)$ where t is time in seconds.

Find the displacement $s(t)$ in metres after 10 seconds given that $s(0) = 0$.

(3 marks)

$$s(t) = \int v(t)dt = \int \big(3 + cos(2\pi t)\big)\,dt$$

$$= 3t + \frac{sin2\pi t}{2\pi} + C$$

Sub. $s(0) = 0. \ \therefore \ \ 0 = 0 + 0 + C.$

$\therefore \; C = 0.$

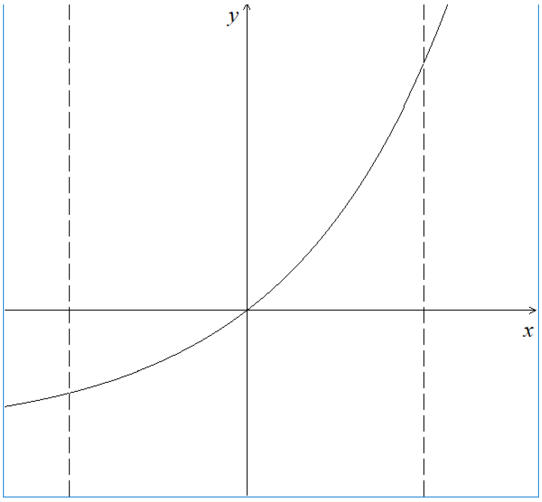
$\therefore \; s(10) = 3 \times 10 + \dfrac{sin(20\pi)}{2\pi} + 0$

$= 30$m.

QUESTION 31

Find the area bounded by $y = e^x - 1$, the x-axis, the line $x = -ln3$ and the line $x = ln3$.

(4 marks)



$x = -ln(3)$ $x = ln(3)$

Area =

$$= -\int_{-ln3}^{0} (e^x - 1)dx + \int_{0}^{ln3} (e^x - 1)dx$$

$$= -[e^x - x]_{-ln3}^{0} + [e^x - x]_{0}^{ln3}$$

$$= -\left((1 - 0) - \left(\frac{1}{3} + ln3\right)\right) + 3 - ln3 - (1 - 0)$$

$$= -\left(\frac{2}{3} - ln3\right) + 2 - ln3$$

$= \frac{4}{3}$ units^2.

QUESTION 32

Half of the cross-section of a speed hump can be modelled with the equation

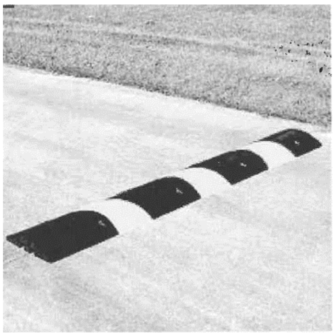
$y = 2\sqrt{\frac{2x}{5}}$ for $x \in [0, 10]$, where x and y are in centimetres.

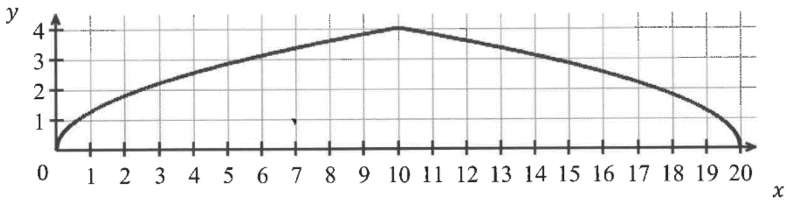
The other half from 10 to 20cm is a reflection of the first function.

The speed hump is to be made of solid concrete.

If there is 16000 cm^3 of concrete available, how long can the speed hump be made?

Show full algebraic working for the integration required.





(4 marks)

$$V = 16000 = \ell \times 2\int_0^{10} 2\sqrt{\frac{2x}{5}}\,dx$$

$$\therefore \frac{4000}{\ell} = \left[\frac{\sqrt{2}(x)^{\frac{3}{2}}}{\frac{3}{2}\times\sqrt{5}}\right]_0^{10}$$

$$\therefore \frac{4000}{\ell} = \left[\frac{\sqrt{2}(10)^{\frac{3}{2}}}{\frac{3}{2}\times\sqrt{5}}\right] - \left[\frac{\sqrt{2}(0)^{\frac{3}{2}}}{\frac{3}{2}\times\sqrt{5}}\right]$$

$$\therefore \frac{4000}{\ell} = \frac{40}{3}$$

$\therefore\ \ell = \frac{12000}{40} = 300$ cm or 3 m.

QUESTION 33

The derivative of the function $f(x)$ is given by $f'^{(x)} = x + k\sqrt{x}$, where $k \in \mathbb{R}$.

$f(x)$ has a tangent at $x = 2$ given by $y = 4x - 3$.

Find the function $f(x)$.

(6 marks)

When $x = 2$, $f'(x) = 4$.

$\therefore \ 4 = 2 + k\sqrt{2}$.

$\therefore \ 2 = k\sqrt{2}$

$k = \frac{2}{\sqrt{2}}$.

$$f(x) = \int f'(x)dx = \frac{x^2}{2} + \frac{2}{\sqrt{2}}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

When $x = 2$, $f(x) = 4 \times 2 - 3 = 5$.

$$\therefore \ 5 = 2 + \frac{2}{\sqrt{2}} \times \frac{2\sqrt{2}}{3/2} + C$$

$$\therefore \ 3 = \frac{8}{3} + C$$

$\therefore \ C = \frac{1}{3}$.

$\therefore \ f(x) = \frac{x^2}{2} + \frac{4x^{\frac{3}{2}}}{3\sqrt{2}} + \frac{1}{3}$.

QUESTION 34

Using standard 6-sided dice, determine the following probabilities:

(a) Rolling a single die and scoring a 6?

(1 mark)

$p = \frac{1}{6}$. $P(X = 6) = \frac{1}{6}$.

(b) Rolling two dice and scoring 5 or more on each one?

(1 mark)

$X \sim \text{Binom}\left(2, \frac{1}{3}\right)$.

$$P(X = 2) = {}^2C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{2-2} = \frac{1}{9}.$$

(c) Rolling three dice and scoring 4 or more on each one?

(1 mark)

$$P(X = 3) = {}^3C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{3-3} = \frac{1}{8}.$$

(d) What is the most likely scenario of the above three results?

(1 mark)

The most likely to occur is (a).

QUESTION 35

A university study plans to model the length of a species of fish in the ocean.

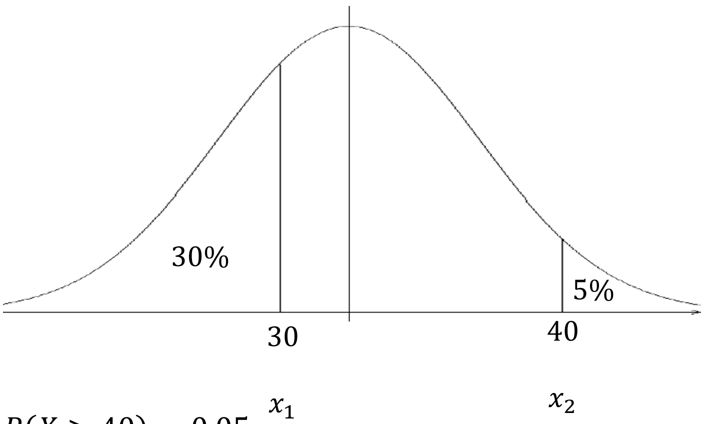
It is assumed that the length of the fish is normally distributed by $X \sim N(\mu, \sigma^2)$.

Fishers are asked to report the percentage of undersize (less than 30 cm) and very large fish (more than 40 cm) they catch.

They report that 30% are undersize and 5% are very large.

Find the estimates of μ and σ for the population of the species in the ocean.

(4 marks)



$P(X > 40) = 0.05$

$\therefore \ \frac{40-\mu}{\sigma} = z_2 = 1.6449.$

$\therefore \ 40 - \mu = 1.6449\sigma$ -Eq1.

$P(X < 30) = 0.3$

$\therefore \ \frac{30-\mu}{\sigma} = z_1 = -0.5244.$

$\therefore \ 30 - \mu = -0.5244\sigma$ -Eq2.

Eq1 – Eq2: $10 = 2.1693\sigma$

$\therefore \ \sigma = 4.6098$

$\therefore \ 40 - \mu = 1.6449 \times 4.6098$

$\therefore \ \mu = 32.4173$ cm.

QUESTION 36

A box contains 20 light globes. Each light globe has a 5% chance of being faulty.

(a) What is the probability that at least 3 light globes are faulty in any one box?

(2 marks)

$X \sim$Faulty globes

$X \sim$Binom$(20\,, 0.05)$

$P(X \geq 3) = 0.07548$.

(b) A packer opens 10 boxes. What is the probability that fewer than 4 boxes have at least 3 light globes that are faulty?

(3 marks)

$Y \sim$Box with at least 3 Faulty globes

$Y \sim$Binom$(10\,, 0.07548)$.

$P(Y < 4) = 0.9953$.

QUESTION 37

A poll is conducted before an election where Candidate A or B is to be elected President.

The poll says that between 45% and 50% of voters will vote for Candidate A with a confidence interval of 95%.

(a) What is the predicted vote for Candidate A, $\hat{p}$?

(1 mark)

$\hat{p} = 0.475$ (halfway between 45% and 50%).

(b) How many people were surveyed to give this confidence interval for the poll result?

(2 marks)

$$M = 0.025 = 1.96\sqrt{\frac{0.475(1-0.475)}{n}}.$$

Solving gives $n = 1532.79$.

So 1533 needed to be surveyed.

(c) Candidate A claims that the true proportion in the population that intend to vote for them is $p = 51\%$.
Assuming that the poll is correct, what would the level of confidence have to be if 51% is to be considered within this interval?

(4 marks)

$\hat{p} = 0.475$, $M = 0.51 - 0.475 = 0.035$.

$$\therefore \quad 0.035 = z\sqrt{\frac{0.475(1-0.475)}{1533}}$$

$$\therefore \quad z = 2.7441$$

Which corresponds to $0.03044 = 0.3044\%$.

Double this area to give 0.6088%.

$100\% - 0.6088\% = 99.3912\%$.

So it is a 99.3912% confidence interval.