Tasmanian Certificate of Education
External Assessment 2020

OFFICE OF TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

PLACE YOUR CANDIDATE LABEL HERE

MATHEMATICS SPECIALISED

(MTS415118)

Time recommended for this paper:

- Working time: 3 hours
- Plus reading time: 15 minutes

Pages:	16
Questions:	30
Attachment:	Information sheet

Candidate Instructions

1. You **MUST** make sure that your responses to the questions in this examination paper will show your achievement in the criteria being assessed.
2. There are **FIVE** sections to this paper.
3. You must answer **ALL** questions.
4. Answer each section in a **SEPARATE** answer booklet.
5. It is recommended that you spend approximately 36 minutes on each section.
6. The Information Sheet for Mathematics Specialised can be used throughout the examination (provided with the paper).
7. No other written material is allowed into the examination.
8. All written responses must be in English.

On the basis of your performance in this examination, the examiners will provide results on the following criteria taken from the course document:

Criterion 4 Solve problems and use techniques involving finite and infinite sequences and series.

Criterion 5 Solve problems and use techniques involving matrices and linear algebra.

Criterion 6 Use differential calculus and apply integral calculus to areas and volumes.

Criterion 7 Use techniques of integration and solve differential equations.

Criterion 8 Solve problems and use techniques involving complex numbers.

Additional Instructions for Candidates

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you show only your answers you will get few, if any, marks.

You are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching and any approved scientific or graphics or CAS calculator (memory may be retained). **Unless instructed otherwise**, calculators may be used to their full capacity when undertaking this examination.

SECTION A

Answer **ALL** questions in this section.

You must show the method and workings you use to solve a question.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 4**.

Section A = 32 marks.

Question 1

Find the n^{th} term of the series whose sum to n terms, S_n, is given by $S_n = 4^n - 1$. (3 marks)

Question 2

Determine $\sum_{r=1}^{n} (r-1)^2$.

Express your answer in fully factorised form. (3 marks)

Question 3

An aeroplane takes off from sea level (0 metres) and ascends to 300m in the first minute.
The rate of height gained then continues to reduce at 97% of the rate of each previous minute.

(a) Determine the maximum height the plane can approach. (2 marks)

(b) Calculate, to the nearest minute, the first time when the plane will be higher than Mt Everest (8848m). (2 marks)

Question 4

(a) Use a method of differences to show that:

$$\sum_{r=1}^{n} \frac{7}{(4r-1)(4r+3)(4r+7)} = \frac{1}{24} - \frac{7}{8(4n+3)(4n+7)} .$$ (5 marks)

(b) Hence evaluate $\sum_{r=1}^{\infty} \frac{7}{(4r-1)(4r+3)(4r+7)}$. (1 mark)

Section A continues.

Section A (continued).

Question 5

Use mathematical induction to show that:

$2\times1+7\times2+14\times2^2+23\times2^3+....+(n^2+2n-1)\times2^{n-1}=n^2\times2^n$. (7 marks)

Question 6

(a) Prove, using formal techniques, that the sequence $\left\{\dfrac{\sin(nx)}{n^2}\right\}$ converges to 0. (3 marks)

(b) Determine whether the following sequences $\{u_n\}$ converge or diverge.

Give brief justifications for your assertions, and state the limits of any sequences that converge.

i. $u_n=\dfrac{2n^2+1}{n(n+1)}$ (2 marks)

ii. $u_n=(-1)^n\left(\dfrac{2n^2+1}{n(n+1)}\right)$ (2 marks)

iii. $u_n=(-1)^n\left(\dfrac{2n+1}{n(n+1)}\right)$ (2 marks)

SECTION B

Answer **ALL** questions in this section.

You must show the method and workings you use to solve a question.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 5**.

Section B = 32 marks.

Question 7

The matrices $Q = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $R = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix}$.

Determine the 2×2 matrix P such that $QP = R$. (3 marks)

Question 8

The circle $x^2 + y^2 = 25$ is transformed with a dilation of factor 2 parallel to the y - axis followed by a shear of factor 3 parallel to the y - axis.

Find the area of the image. (4 marks)

Question 9

Any 2×2 matrix A is called *involutory* if it satisfies $A^2 = I$ where I is the identity matrix.

(a) Determine the condition for the matrix $B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ to be involutory. (2 marks)

(b) Determine the possible values for the determinant of the matrix A. (3 marks)

Question 10

Two planes are defined by the equations:

$$x + 2y + 3z = 4$$

and $4x + 3y - 2z = 1$

Find a parametric representation of the line of intersection. (4 marks)

Section B continues.

Section B (continued).

Question 11

A function is rotated anti-clockwise through $\frac{\pi}{4}$ radians about the origin and then reflected in the line $y = x$. The equation of the image obtained under this composition of transformations is $(x+y)^2 - \sqrt{2}\left(x-y\right) - 8 = 0$.

(a) Determine the composite matrix for the transformations. (2 marks)

(b) Hence show that the original (source) equation is $y = x^2 - 4$. (3 marks)

(c) On the same set of axes, sketch graphs of the source and the image. (2 marks)

Question 12

(a) State the augmented matrix, in terms of k, that represents the following equations: (1 mark)

$$x + y + kz = 0$$

$$3x - 7y + z = 0$$

$$kx - 8y - z = 0$$

(b) Use Gauss-Jordan elimination techniques to transform the matrix in (a) into the reduced row echelon form $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. (5 marks)

(c) Determine all values for k such that the equations have solutions other than $x = y = z = 0$.

(3 marks)

SECTION C

Answer **ALL** questions in this section.

You must show the method and workings you use to solve a question.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 6**.

Section C = 32 marks.

Question 13

(a) If $y = \log_2 x^2,\ x \neq 0$, show that $\frac{dy}{dx} + x\left(\frac{d^2y}{dx^2}\right) = 0$. (3 marks)

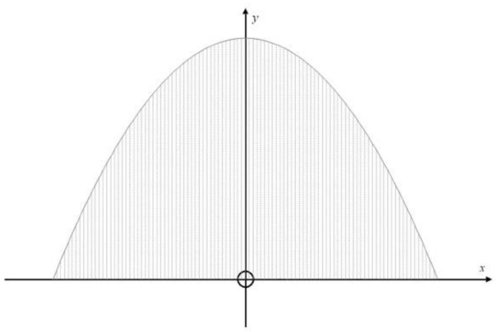
(b) If $y = \frac{\arctan x}{x}$, evaluate $\frac{dy}{dx}$ at $x = 1$. (3 marks)

Question 14

Find the equation of the tangent to the curve $e^{x-y} - x^2 y = 0$ at the point $(1,1)$. (5 marks)

Question 15

The shape of a window is defined by the function $y = \frac{5}{2}\left[2\cos^2\left(\frac{\pi x}{4}\right) - 1\right]$.



Determine the shaded area of the window as shown in the diagram above. (5 marks)

Section C continues.

Section C (continued).

Question 16

Show that the function $f(x)=\dfrac{x^2}{\sqrt{x-1}}$ has a local minimum at $x=\dfrac{4}{3}$. (4 marks)

Question 17

Consider the function $f(x)=\dfrac{4}{\sqrt{x^2+4}}$.

(a) Show that $f(x)$ has a point of inflection at $\left(\sqrt{2},\dfrac{2\sqrt{6}}{3}\right)$. (4 marks)

(b) The region enclosed by the graph of $y=f(x)$ between the values of $x=0$ and $x=2\sqrt{3}$ is rotated completely about the x - axis.

Find the exact value of the volume of the solid of revolution that is formed. (3 marks)

Question 18

The area enclosed by the functions $y=(x-1)^2$ and $y=\sqrt{x-1}$ is rotated completely around the y - axis.

Prove that the solid formed has a volume of $\dfrac{29\pi}{30}$. (5 marks)

SECTION D

Answer **ALL** questions in this section.

You must show the method and workings you use to solve a question.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 7**.

Section D = 32 marks.

Question 19

Find:

(a) $\int \frac{\cos x}{1+\sin^2 x}\, dx$ (3 marks)

(b) $\int_1^{\sqrt{2}} \frac{1}{\sqrt{16-4x^2}}\, dx$ (3 marks)

Question 20

Show that $\int_0^1 x^2 e^{-2x}\, dx = \frac{1}{4}\left[1-5e^{-2}\right]$. (4 marks)

Question 21

Evaluate $\int_0^1 \frac{x^5}{x^3+1}\, dx$. (4 marks)

Question 22

Solve the differential equation $\frac{dy}{dx} = \frac{y^2-4}{2}$, given that $y=3$ when $x=0$. (6 marks)

Question 23

Find the general solution to the homogeneous differential equation $\frac{dy}{dx} = \frac{3y^2-x^2}{2xy}$. (6 marks)

Section D continues.

Section D (continued).

Question 24

Above sea level, the atmospheric pressure (P) decreases with an increase in altitude (A), and can be modelled by the differential equation

$$\frac{dP}{dA} = kP \text{ where } k \text{ is a constant.}$$

(a) Explain why k is negative. (1 mark)

(b) When $A = 0 \text{ m}$, then $P = 1013 \text{ hPa}$ and when $A = 1220 \text{ m}$, then $P = 875 \text{ hPa}$.
Determine the value of k accurate to 5 decimal places.

(4 marks)

(c) Determine, to the nearest metre, the altitude at which the atmospheric pressure is 500 hPa. (1 mark)

SECTION E

Answer **ALL** questions in this section.

You must show the method and workings you use to solve a question.

Use a **SEPARATE** answer booklet for this section.

This section assesses **Criterion 8**.

Section E = 32 marks.

Question 25

Given that $z=\sqrt{2}+i$, $w=\frac{iz}{\sqrt{2}}$ and $v=\overline{w}$, represent the points z , w and v on an Argand diagram.

(3 marks)

Question 26

(a) If $z=\frac{3+2i}{4-3i}$, find the real and imaginary components of $\frac{1}{z}$. (3 marks)

(b) Find a (>0) and b such that $(a+ib)=\sqrt{5+12i}$. (3 marks)

Question 27

(a) Solve $z^6=-64$. (3 marks)

(b) Hence, factorise $f(z)=z^6+64$ into **THREE** real, quadratic factors. (3 marks)

Question 28

The polynomial $P(z)=z^3-7z^2+az+b;\ a,\ b\in\mathbb{R}$ has a root of $3-2i$.

(a) Find the values of a and b. (3 marks)

(b) Factorise $P(z)$ into **THREE** linear factors. (3 marks)

Section E continues.

Section E (continued).

Question 29

Show the region on an Argand diagram that is defined by:

$$\left\{\left[\operatorname{Re}(z-3-5i)\right]^2 < \operatorname{Im}(z)+2\right\} \cap \left\{\operatorname{Im}(z-5+2i) < 1\right\} \cap \left\{Arg(z-2+i) \geq -\frac{\pi}{4}\right\}.$$

Label all relevant intersection points. (7 marks)

Question 30

Given $z = 1 + i$, determine **ALL** real values of n such that $z^n + (\bar{z})^n = 0$. (4 marks)

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